

On the instability problem of a 3-D transonic oblique shock wave

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Abstract

In this paper, we are concerned with the instability problem of a 3-D transonic oblique shock wave for the steady supersonic flow past an infinitely long sharp wedge. The flow is assumed to be isentropic and irrotational. It was indicated in pages 317 of [9] that if a steady supersonic flow comes from minus infinity and hits a sharp symmetric wedge, then it follows from the Rankine-Hugoniot conditions and the physical entropy condition that there possibly appear a weak shock or a strong shock attached at the edge of the sharp wedge, which corresponds to a supersonic shock or a transonic shock, respectively. The question arises which of the two actually occurs. It has frequently been stated that the strong one is unstable and that, therefore, only the weak one could occur. However, a convincing proof of this instability has apparently never been given. The aim of this paper is to understand such a longstanding open question. We will show that the attached 3-D transonic oblique shock problem is overdetermined, which implies that the 3-D transonic shock is **unstable** in general.

Keywords: Supersonic flow, potential equation, transonic oblique shock, modified Bessel function, overdetermined, unstable

Mathematical Subject Classification 2000: 35L70, 35L65, 35L67, 76N15

§1. Introduction

In this paper, we are concerned with the instability problem of a 3-D transonic oblique shock for the steady supersonic flow past an infinitely long sharp wedge (see Figure 1 below). As indicated in pages 317 of [9]: if a supersonic steady flow comes from minus infinity and hits a sharp symmetric wedge, then it follows from the Rankine-Hugoniot

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conditions and the physical entropy condition that there will appear a weak shock or a strong shock attached at the edge of the sharp wedge, which corresponds to a supersonic shock or a transonic shock, respectively. The question arises which of the two shocks actually occurs. It has frequently been stated that the strong one is unstable and that, therefore, only the weak one could occur. However, a convincing proof of this instability has apparently never been given. The aim of this paper is to understand such a long-standing open question. With respect to the 2-D weak oblique shock, under some different assumptions on the 2-D sharp wedge, the authors in [4, 19, 23, 32] have respectively established the local or global existence and stability of a supersonic shock solution or a weak solution for the perturbed supersonic incoming flow past a 2-D sharp curved wedge. For the 3-D weak oblique shock, Chen S.X. in [5] has shown its local stability. With respect to the 2-D strong oblique shock, under certain pressure condition at infinity in the downstream subsonic region, the authors in [6] and [33] have proved the global existence and stability of a transonic shock for the 2-D potential equation and the 2-D full Euler system respectively, which are contrary with the conjecture on the instability of the transonic oblique shock (this instability conjecture has been mentioned in the above). In addition, for the 2-D unsteady potential equation, the authors in [12] constructed a self-similar analytic solution which connects an attached 2-D strong shock and an attached 2-D weak shock when a supersonic flow hits a 2-D sharp wedge. Note that the realistic world is three-dimensional. The aim of this paper is to show that the attached 3-D transonic shock problem is overdetermined, which means that the 3-D transonic shock is **unstable** in general and further gives a rather positive illustration on the instability of a 3-D transonic oblique shock. This also indicates that the space dimensions are essential for answering the stability or instability of the transonic oblique shocks.

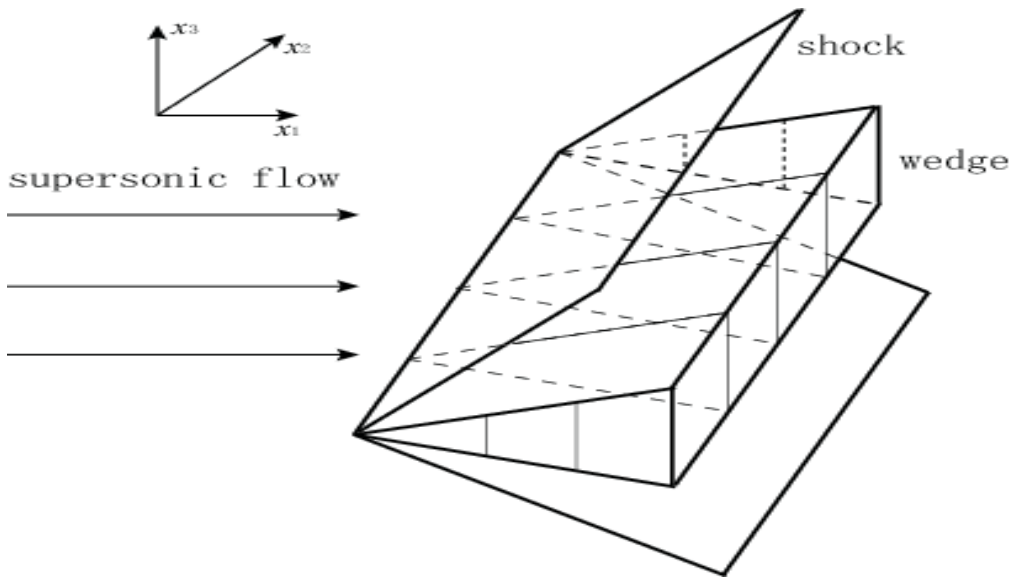


Figure 1. A uniform supersonic flow past a 3-D sharp wedge

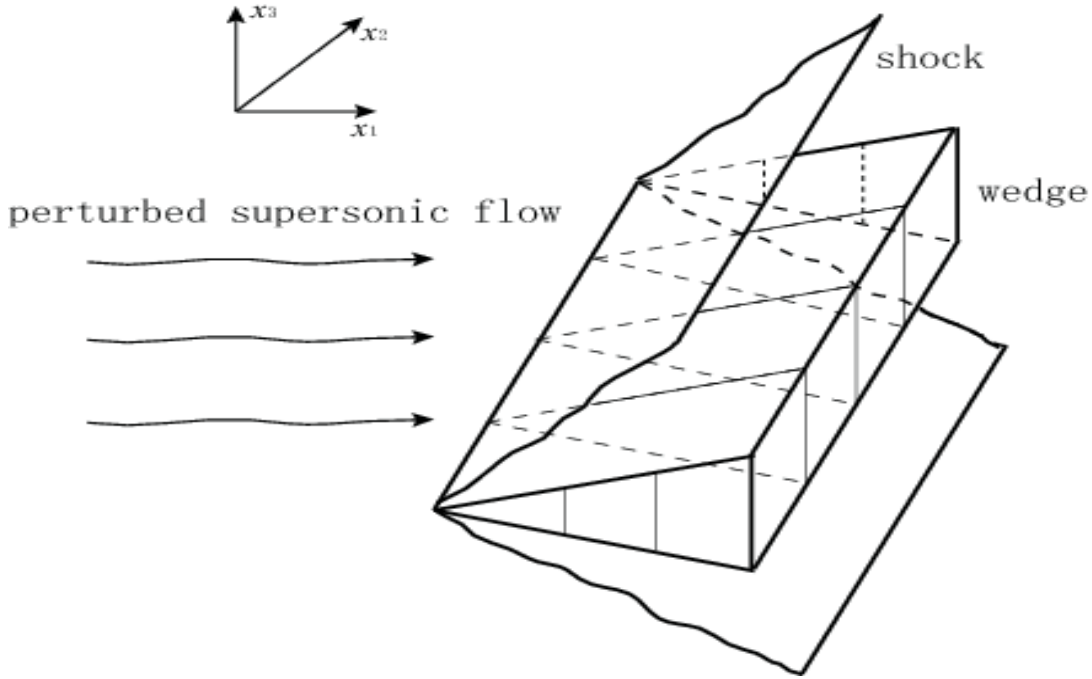


Figure 2. A perturbed supersonic flow past a 3-D sharp wedge

We will assume that the supersonic incoming flow is of a small perturbation with respect to the constant supersonic state $(\rho_0, q_0, 0, 0)$ and such a flow hits the sharp 3-D wedge $\{x : x_1 \geq 0, x_2 \in \mathbb{R}, -b_0x_1 \leq x_3 \leq b_0x_1\}$ along the x_1 -direction (see Figure 2 above). Due to the non-interaction property of the transonic oblique shocks on two sides of the wedge, then it suffices to consider our transonic shock problem only in the upper half-space $x_3 \geq 0$ and use a ramp $\{x : x_1 \in \mathbb{R}, x_2 \in \mathbb{R}, 0 \leq x_3 \leq b_0x_1\}$ instead of the wedge (see Figure 3 below).

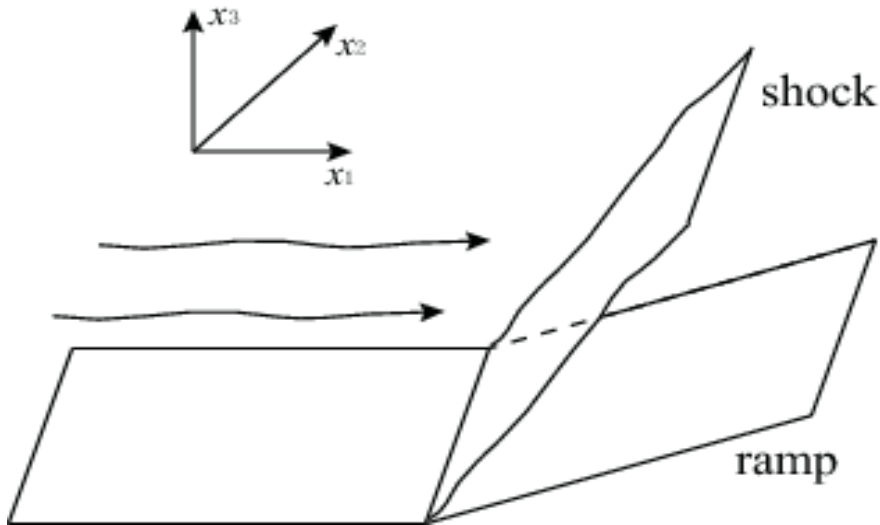


Figure 3. A perturbed supersonic flow past a 3-D ramp

The steady and compressible 3-D Euler system is described by

$$\begin{cases} \sum_{j=1}^3 \partial_j(\rho u_j) = 0, \\ \sum_{j=1}^3 \partial_j(\rho u_i u_j) + \partial_i P = 0, \quad i = 1, 2, 3, \end{cases} \quad (1.1)$$

where $\rho > 0$ is the density, $u = (u_1, u_2, u_3)$ is the velocity, and $P = A\rho^\gamma$ ($1 < \gamma < 3$) is the pressure with $A > 0$ a fixed constant. In addition, $c(\rho) = \sqrt{P'(\rho)}$ is called the local sound speed.

In our paper, we will use the potential equation to describe the motion of the gas (this model or its variant models have been applied in many other transonic or supersonic shock problems, one can see [2-3], [13], [17], [25], [34] and so on). Let $\varphi(x)$ be the potential of velocity $u = (u_1, u_2, u_3)$, i.e., $u_i = \partial_i \varphi$, then it follows from the Bernoulli's law that

$$\frac{1}{2}|\nabla_x \varphi|^2 + h(\rho) = C_0, \quad (1.2)$$

here $\nabla_x = (\partial_1, \partial_2, \partial_3)$, $h(\rho) = \frac{c^2(\rho)}{\gamma - 1}$ is the specific enthalpy, and $C_0 = \frac{1}{2}q_0^2 + h(\rho_0)$ is the Bernoulli's constant which is determined by the uniform supersonic incoming flow from the minus infinity with the constant velocity $(q_0, 0, 0)$ and the constant density $\rho_0 > 0$ (see Figure 1 above).

By (1.2) and the implicit function theorem, the density function $\rho(x)$ can be expressed as

$$\rho = h^{-1}(C_0 - \frac{1}{2}|\nabla_x \varphi|^2) \equiv H(\nabla_x \varphi). \quad (1.3)$$

Substituting (1.3) into the mass conservation equation $\sum_{j=1}^3 \partial_j(\rho u_j) = 0$ in (1.1) yields

$$\sum_{i=1}^3 ((\partial_i \varphi)^2 - c^2) \partial_i^2 \varphi + 2 \sum_{1 \leq i < j \leq 3} \partial_i \varphi \partial_j \varphi \partial_{ij}^2 \varphi = 0, \quad (1.4)$$

where $c = c(H(\nabla_x \varphi))$.

Suppose that the disturbed velocity potentials before and behind the possible attached shock front $x_3 = \chi(x_1, x_2)$ with $\chi(0, x_2) = 0$ are denoted by $\varphi^-(x)$ and $\varphi^+(x)$ respectively. In this case, the system (1.4) can be split into two equations, that is, $\varphi^\pm(x)$ satisfy the following equations in the corresponding domains

$$\begin{aligned} \sum_{i=1}^3 ((\partial_i \varphi^-)^2 - (c^-)^2) \partial_i^2 \varphi^- + 2 \sum_{1 \leq i < j \leq 3} \partial_i \varphi^- \partial_j \varphi^- \partial_{ij}^2 \varphi^- &= 0 \\ \text{in } \{x_1 > 0, x_3 > \chi(x_1, x_2)\} \text{ or } \{x_1 \leq 0\} & \end{aligned} \quad (1.5)$$

and

$$\sum_{i=1}^3 ((\partial_i \varphi^+)^2 - (c^+)^2) \partial_i^2 \varphi^+ + 2 \sum_{1 \leq i < j \leq 3} \partial_i \varphi^+ \partial_j \varphi^+ \partial_{ij}^2 \varphi^+ = 0 \quad \text{in } \{x_1 > 0, x_3 < \chi(x_1, x_2)\} \quad (1.6)$$

with $c^\pm = c(\rho^\pm) = c(H(\nabla_x \varphi^\pm))$.

It is easy to verify that (1.5) is strictly hyperbolic with respect to x_1 for $\partial_1 \varphi^- > c^-$ and (1.6) is strictly elliptic for $|\nabla_x \varphi^+| < c^+$.

On the ramp surface $\Sigma : x_3 = b_0 x_1$, φ^+ satisfies

$$b_0 \partial_1 \varphi^+ - \partial_3 \varphi^+ = 0 \quad \text{on } \Sigma. \quad (1.7)$$

Meanwhile, on the possible transonic shock surface $\Gamma : x_3 = \chi(x_1, x_2)$ with $\chi(0, x_2) = 0$, the Rankine-Hugoniot condition is

$$[H \partial_1 \varphi] \partial_1 \chi + [H \partial_2 \varphi] \partial_2 \chi - [H \partial_3 \varphi] = 0 \quad \text{on } \Gamma, \quad (1.8)$$

here we especially point out that the condition $\chi(0, x_2) = 0$ comes from the assumption that the transonic shock is attached at the edge of ramp.

Moreover, the potential $\varphi^+(x)$ is continuous across the shock surface Γ , namely,

$$\varphi^+(x_1, x_2, \chi(x_1, x_2)) = \varphi^-(x_1, x_2, \chi(x_1, x_2)), \quad (1.9)$$

which obviously means

$$\varphi^+(0, x_2, 0) = \varphi^-(0, x_2, 0). \quad (1.10)$$

On Γ , it follows from the physical entropy condition that

$$\rho^-(x_1, x_2, \chi(x_1, x_2)) < \rho^+(x_1, x_2, \chi(x_1, x_2)). \quad (1.11)$$

In addition, the stable subsonic velocity field behind Γ will admit a determined state:

$$|\nabla_x \varphi^+| < c^+, \quad \text{and} \quad \lim_{x_1 \rightarrow +\infty} \nabla_x \varphi^+(x) \text{ exists for } b_0 x_1 \leq x_3 \leq \chi(x_1, x_2). \quad (1.12)$$

Finally, we pose the following perturbed initial conditions with respect to the uniform supersonic constant flow $(\rho_0, q_0, 0, 0)$

$$\varphi^-(0, x_2, x_3) = \varepsilon \varphi_0^-(x_2, x_3), \quad \partial_1 \varphi^-(0, x_2, x_3) = q_0 + \varepsilon \varphi_1^-(x_2, x_3), \quad (1.13)$$

where $\varepsilon > 0$ a small constant, $\varphi_i^-(x_2, x_3) \in C^\infty(\mathbb{R}^2)$ ($i = 0, 1$) are supported in $(0, l)$ with respect to the variable x_3 , and $l > 0$ is some fixed positive number, moreover, $\varphi_i^-(x_2, x_3) = \varphi_i^-(x_2 + 2\pi, x_3)$ holds for $i = 0, 1$. Here we point out that these assumptions on $\varphi_i^-(x_2, x_3)$ ($i = 0, 1$) do not lose the generality by the finite propagation speed property for the hyperbolic equation (1.5) (one can see more illustrations in Remark 1.3 below).

In order to solve the transonic shock problem (1.5)-(1.6) together with (1.7)-(1.13), we will use the partial hodograph transformation in [22] or [26-27] to fix the free boundary Γ .

To this end, we set $\Phi(x) = \varphi^-(x) - \varphi^+(x)$ and then it follows from a direct computation that the problem (1.6)-(1.12) can be rewritten as

$$\left\{ \begin{array}{ll} \sum_{i,j=1}^3 a_{ij}(\nabla_x \varphi^- - \nabla_x \Phi) \partial_{ij}^2 \Phi = \sum_{i,j=1}^3 a_{ij}(\nabla_x \varphi^- - \nabla_x \Phi) \partial_{ij}^2 \varphi^-, & \\ \Phi(x_1, x_2, \chi(x_1, x_2)) = 0 & \text{on } \Gamma, \\ ((\rho_+ - \rho_-) \partial_1 \varphi^- - \rho_+ \partial_1 \Phi) \partial_1 \chi + ((\rho_+ - \rho_-) \partial_2 \varphi^- - \rho_+ \partial_2 \Phi) \partial_2 \chi \\ \quad - ((\rho_+ - \rho_-) \partial_3 \varphi^- - \rho_+ \partial_3 \Phi) = 0 & \text{on } \Gamma, \\ \partial_3 \Phi - b_0 \partial_1 \Phi = \partial_3 \varphi^- - b_0 \partial_1 \varphi^- & \text{on } \Sigma, \\ \Phi(x_1, x_2 + 2\pi, x_3) = \Phi(x), & \\ \lim_{x_1 + x_3 \rightarrow +\infty} \nabla_x \Phi \text{ exists,} & \end{array} \right. \quad (1.14)$$

where $\varphi^-(x)$ is the potential of the supersonic incoming flow, which can be shown to be extended across the shock Γ (see Lemma 2.4 and Remark 2.1 in §2 below), and

$$a_{ii}(\nabla_x \varphi^+) = 1 - \frac{(\partial_i \varphi^+)^2}{(c^+)^2}, \quad i = 1, 2, 3, \quad a_{ij}(\nabla_x \varphi^+) = -\frac{\partial_i \varphi^+ \partial_j \varphi^+}{(c^+)^2}, \quad 1 \leq i \neq j \leq 3.$$

As in [26-27], we introduce the following partial hodograph transformation to fix the shock surface Γ

$$\left\{ \begin{array}{l} y_1 = \frac{\Phi(x)}{q_0}, \\ y_2 = x_2, \\ y_3 = \frac{b_0 \Phi(x)}{q_0} + x_3 - b_0 x_1, \end{array} \right. \quad (1.15)$$

In this case, the shock surface Γ is changed into $y_1 = 0$. Suppose that the inverse transformation of (1.15) is denoted by

$$\left\{ \begin{array}{l} x_1 = u(y), \\ x_2 = y_2, \\ x_3 = y_3 - b_0 y_1 + b_0 u(y), \end{array} \right. \quad (1.16)$$

where the definition domain of $u(y)$ is the open domain $Q = \{y \in \mathbb{R}^3 : y_1 > 0, y_2 \in \mathbb{R}, y_3 > b_0 y_1\}$. With respect to the validity of the invertibility for the transformation (1.15), one can see the detailed illustrations in §3 below. In addition, it follows from (1.9) and (1.16) that

$$u(0, y_2, 0) = 0. \quad (1.17)$$

By (1.14) and (1.16)-(1.17) together with a direct computation, we have

$$\left\{ \begin{array}{l} L(u, \nabla_y u, \nabla_y^2 u) \\ \equiv \sum_{1 \leq i \leq j \leq 3} A_{ij}(u, \nabla_y u) \partial_{y_i y_j}^2 u + \frac{1}{q_0} \sum_{1 \leq i \leq j \leq 3} a_{ij}(\nabla_x \varphi^- - \nabla_x \Phi) \partial_{x_i x_j}^2 \varphi^- = 0 \\ G_1(u, \nabla_y u) = 0 \\ G_2(u, \nabla_y u) = 0 \\ u(0, y_2, 0) = 0, \\ u(y_1, y_2 + 2\pi, y_3) = u(y), \\ \lim_{y_1 + y_3 \rightarrow +\infty} |\nabla_y u| \text{ exists,} \end{array} \right. \quad \begin{array}{l} \text{in } Q, \\ \\ \text{on } y_3 = b_0 y_1, \\ \text{on } y_1 = 0, \\ \\ \\ \end{array} \quad (1.18)$$

where the concrete expressions of $A_{ij}(u, \nabla_y u)$, $G_1(u, \nabla_y u)$ and $G_2(u, \nabla_y u)$ will be given in §3 below.

Therefore, solving the transonic shock problem (1.5)-(1.6) together with (1.7)-(1.13) is completely equivalent to solving (1.18). However, unfortunately, (1.18) is an overdetermined problem due to the restriction $u(0, y_2, 0) = 0$ for all $y_2 \in \mathbb{R}$. More precisely, the following problem can be shown to be uniquely solvable for any fixed $y_2^0 \in \mathbb{R}$

$$\left\{ \begin{array}{l} L(u, \nabla_y u, \nabla_y^2 u) = 0 \\ G_1(u, \nabla_y u) = 0 \\ G_2(u, \nabla_y u) = 0 \\ u(0, y_2^0, 0) = 0, \\ u(y_1, y_2 + 2\pi, y_3) = u(y), \\ \lim_{y_1 + y_3 \rightarrow +\infty} \nabla_y u \text{ exists.} \end{array} \right. \quad \begin{array}{l} \text{in } Q, \\ \\ \text{on } y_3 = b_0 y_1, \\ \text{on } y_1 = 0, \\ \\ \end{array} \quad (1.19)$$

Here we emphasize that the difference between (1.18) and (1.19) is: only $u(0, y_2^0, 0) = 0$ holds for some fixed point $(0, y_2^0, 0)$ in (1.19) other than $u(0, y_2, 0) = 0$ holds in (1.18) for all $y_2 \in \mathbb{R}$.

We now state our main result in this paper.

Theorem 1.1. Assume that $b_0 > 0$ is a small constant, namely, the 3-D ramp is sharp, then for suitably large supersonic incoming speed q_0 , the nonlinear problem (1.19) admits a unique smooth solution $u(y)$ in Q , which illustrates that (1.18) is overdetermined.

Remark 1.1. The detailed descriptions on the regularities of $u(y)$ in Theorem 1.1 will be given in Theorem 3.1 of §3 below.

Remark 1.2. By the overdetermination of the transonic shock problem (1.5)-(1.6) together with (1.7)-(1.13) in Theorem 1.1, we know that the transonic oblique shock is unstable in general. If one could find another point $(0, y_2^1, 0) \neq (0, y_2^0, 0)$ such that $u(0, y_2^1, 0) \neq 0$ holds for the solution u to (1.19), then the conjecture of the instability for the attached transonic oblique shock is verified in case of the potential flow equation.

Remark 1.3. Although we pose some restrictions on the perturbed initial data $\varphi_i^-(x_2, x_3)$

($i = 0, 1$) in (1.13), this does not lose the generality. Indeed, if $\varphi_i^-(x_2, x_3) \in C_0^\infty(\mathbb{R} \times \mathbb{R}^+)$, then we can take the smooth initial data $\varphi_{i,L}^-(x_2, x_3)$ with a period L for the variable x_2 instead of $\varphi_i^-(x_2, x_3)$ in (1.13), where $L > 1$ is any fixed constant and $\varphi_{i,L}^-(x_2, x_3) = \varphi_i^-(x_2, x_3)$ holds for $x_2 \in [0, L]$. In this case, the related problem (1.19) on $u_L(y)$ can be solved by Theorem 1.1. Moreover, it follows from the proof procedure of Theorem 1.1 that all the $u_L(y)$ for $L \geq 1$ are uniformly bounded for $y \in [0, \infty) \times K \times [0, \infty)$, here K is any fixed compact set in \mathbb{R} . Subsequently, letting $L \rightarrow \infty$, then (1.19) can be solved for the given initial data $\varphi_i^-(x_2, x_3)$ ($i = 0, 1$).

Since the oblique shocks and the conic shocks are two kinds of basic attached shocks for the supersonic flows past the sharp wedges or sharp cones, we now comment on some interesting and systematic results on the attached conic shocks. It was indicated in pages 317-318 and 414 of [9] that if a uniform supersonic steady flow hits a sharp cone in direction of its axis, then it follows from the Rankine-Hugoniot conditions and the physical entropy condition that there possibly occur a weak conic shock (see Figure 4 below) and a strong conic shock (see Figure 5 below) attached at the tip of the cone (this physical phenomena is completely similar to that for the steady supersonic flow past a sharp wedge). For the potential equation, under various assumptions on the supersonic incoming flows and the sharp vertex angles of the conic bodies, the authors have established the local or global existence and stability of the weak conic shocks or strong conic shocks, one can see [7-11], [17-18], [21], [25-27] and the references therein. For the full Euler system, because of the essential influences of the rotations, the authors in [30] and [28] have shown the nonexistence of the global weak solution with only one stable weak conic shock and the instability of a global transonic conic shock for the steady supersonic flow past a sharp conic body, respectively. Therefore, these results have given a basic answer for the global stability or instability of weak and strong conic shocks.

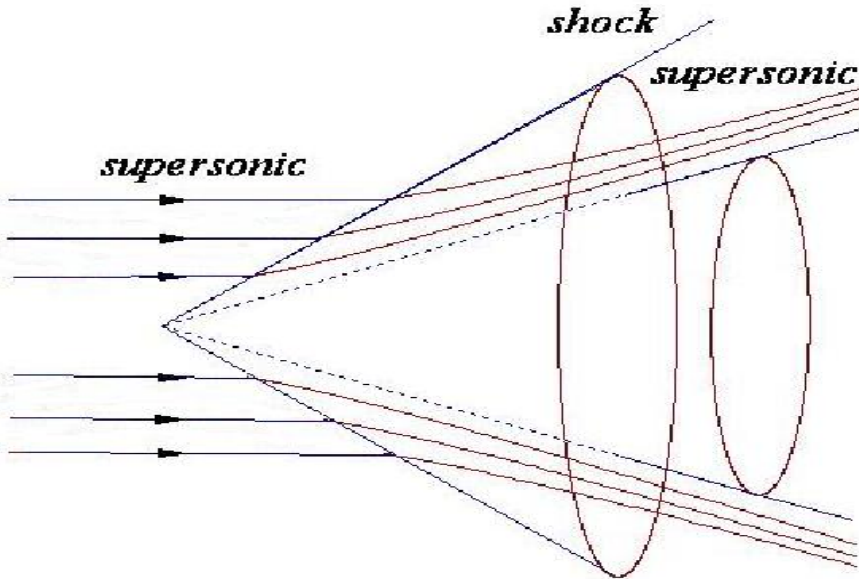


Figure 4. A supersonic shock for the supersonic flow past a 3-D sharp cone

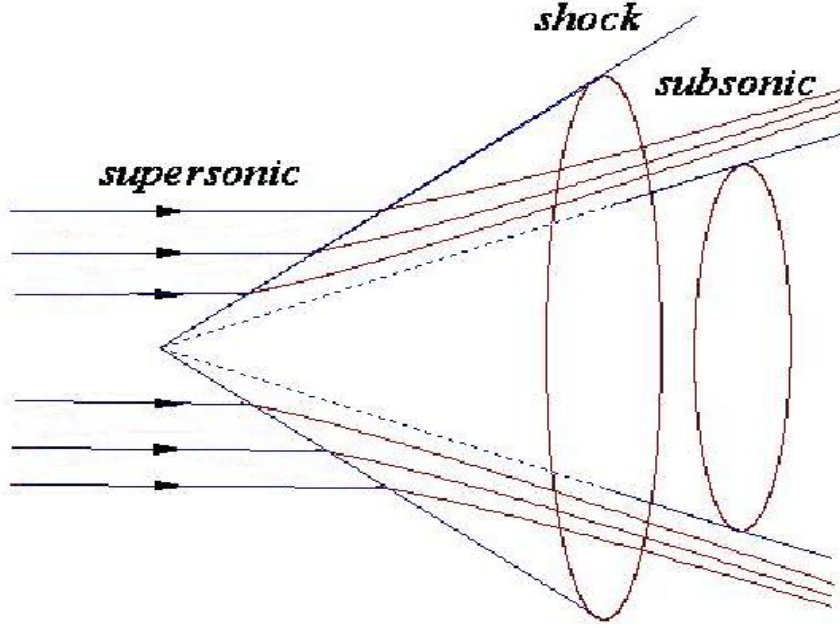


Figure 5. A transonic shock for the supersonic flow past a 3-D sharp cone

We now mention some transonic shock problems studied recently in [2-3], [6], [12-13], [26-27], [33-34] and the references therein. In these papers, the considered domains are either 2-D polygons or 3-D conic bodies. For the 2-D polygon domains (see [2-3], [6], [12-13], [33-34]), it follows from the maximum principle and the barrier function method for the second order elliptic equations in the 2-D irregular regions that one can obtain at least $C^{1,\alpha}$ ($0 < \alpha < 1$) regularities of the corresponding shock curves and the downstream subsonic solutions. The $C^{1,\alpha}$ regularity is crucial in studying the free boundary problem on the second order nonlinear elliptic equations whose coefficients contain the gradients of solution. For the 3-D conic domains (see [26-27], where the maximum principle can not be used directly), by utilizing the Sturm-Liouville theorem and separation variable method, we can write out the expression of the solution to the linearized elliptic equations and the corresponding boundary conditions, subsequently we can obtain the regularity, existence and a priori estimates of the solution to the nonlinear problem in the conic domain and the suitable weighted Hölder space with two different weights near the conic point and at infinity. However, it seems rather difficult for us to choose a weighted Hölder space as in [26-27] to deal with the corresponding linearized equation of (1.19) in the 3-D unbounded wedge domain. The reason is that: we can not expect the solution u of (1.19) to satisfy $u(0, y_2, 0) \equiv 0$ for all $y_2 \in \mathbb{R}$, thus such properties of $|u(y)| \leq Cy_1^{\delta_0}$ ($\delta_0 > 0$) near the edge $y_1 = y_3 = 0$ and $|u(y)| \leq Cy_1^{-\delta_1}$ ($\delta_1 > 0$) for sufficiently large $y_1 > 0$ can not hold simultaneously. Note that such kind of weighted space in [26-27] is crucial in deriving the solvability of the related linearized potential equation in the unbounded conic domain by the separation variable method. Therefore, in this paper we should use some other ingredients to overcome this difficulty so that our problem (1.19) in the unbounded wedge region can be treated.

Next we comment on the proof of Theorem 1.1. To solve (1.19), we will linearize the nonlinear problem on u by use of the largeness of q_0 and the detailed properties on the background solution, here the so-called background solution is referred as one to the problem (1.19) when the uniform supersonic steady flow $(\rho_0, q_0, 0, 0)$ hits the ramp $\{x : x_1 > 0, x_2 \in \mathbb{R}, 0 < x_3 < b_0 x_1\}$ along x_1 -direction. By the linearization, we essentially obtain the Laplacian equation $\Delta v = f$ in \mathbb{R}^3 with two Neumann boundary conditions on two different planes in an angular region, a vanishing condition of the first order derivatives Dv at infinity and a restriction condition $v(0, 0, 0) = 0$ (one can see (4.1) in §4 below). To study the solvability, regularity of v and derive the a priori estimates of v in the unbounded wedge region, at first we will restrict our linearized problem in a bounded wedge domain in addition a Neumann-type boundary condition on the cut-off surface (see (4.4) of §4). In this case, by use of Sturm-Liouville theorem, the separation variable method, we can derive the concrete expression of the solution v_L to the cut-off problem (4.4). It follows from the detailed estimates on the related eigenvalues and eigenfunctions that we can get the existence and $C^{1,\delta}$ ($0 < \delta < 1$) regularity of v_L up to the boundaries (including the two boundaries of the angular domain) in Lemma 5.2 of §5. Based on these crucial estimates and the scaling techniques for the linear elliptic equations, we can obtain the global estimates of v in the whole wedge domain by taking the limit $L \rightarrow \infty$ for v_L . Finally, by taking a suitable iteration scheme and using the largeness of q_0 and the uniform estimates on the solution to the linearized problem, we can complete the proof on Theorem 1.1.

Our paper is organized as follows. In §2, at first, we give some useful information on the background solution for large q_0 , which essentially corresponds to a 2-D transonic oblique shock solution for the uniform supersonic flow past a 2-D sharp wedge. Secondly, we will define some weighted Hölder spaces which will be used in subsequent sections. Thirdly, we list or derive some basic properties of the modified Bessel functions of the first and second kind of order ν ($\nu \in \mathbb{R}$) so that one can use the separation method to study our problems in subsequent §3-§6. Fourthly, a global solvability on the problem (1.5) with (1.13) near the shock Γ is given. In §3, we will reformulate the problem (1.5)-(1.6) together with (1.7)-(1.13) into (1.18) meanwhile the detailed expressions of the coefficients in (1.18) can be given. Moreover, a more precise description on Theorem 1.1 in the weighted Hölder space will be given in Theorem 3.1. In §4, the linearized equation and boundary conditions of (1.19) are given in (4.1), subsequently, a cut-off problem (4.4) with a suitable Neumann boundary condition on the cut-off surface $\sqrt{y_1^2 + y_3^2} = L$ is studied in details, where the solvability of (4.4) and the rough regularity of the solution v_L to (4.4) in related weighted Hölder space are shown. In §5, the higher regularities of v_L in (4.4) are obtained by the classical Schauder estimate and the regularity theory of solutions to the second order elliptic equations in a 3-D bounded angular region. Moreover, the global solvability and estimates of the solution to (4.1) in the unbounded angular domain Q are established. In §6, the uniqueness of solution u to (4.1) is proved by the separation variable method other than by the usual maximum principle for the second order elliptic equations since it seems that there is no maximum principle for the problem (4.1) due to the 3-D unbounded angular region and the Neumann boundary conditions (note that u and $\nabla_y u$ are actually unknown on the edge $\{y_1 = y_3 = 0\}$ of Q). Based on the estimates

in §4-§6, Theorem 3.1 and further Theorem 1.1 can be shown in §7. In addition, some complicated and useful computations are carried out in the Appendix.

In what follows, we will use the following conventions:

For large q_0 , $O(q_0^{-\nu})$ ($\nu > 0$) denotes a bounded quantity such that $|O(q_0^{-\nu})| \leq Cq_0^{-\nu}$, where $C > 0$ is a generic positive constant.

The Gamma function $\Gamma(a)$ for $a > 0$ and the Beta function $B(a, b)$ ($a, b > 0$) are respectively defined as

$$\Gamma(a) = \int_0^{+\infty} t^{a-1} e^{-t} dt \quad \text{for } a > 0,$$

$$B(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt \quad \text{for } a > 0, b > 0.$$

§2. Some preliminaries

At first, we study the background solution to (1.5)-(1.6) together with (1.7)-(1.13) and derive some useful properties of the transonic oblique shock for the uniform supersonic incoming flow past a sharp ramp. Since $u_2 = 0$ always holds in the background solution, it is only required to consider a 2-D transonic oblique shock problem temporarily.

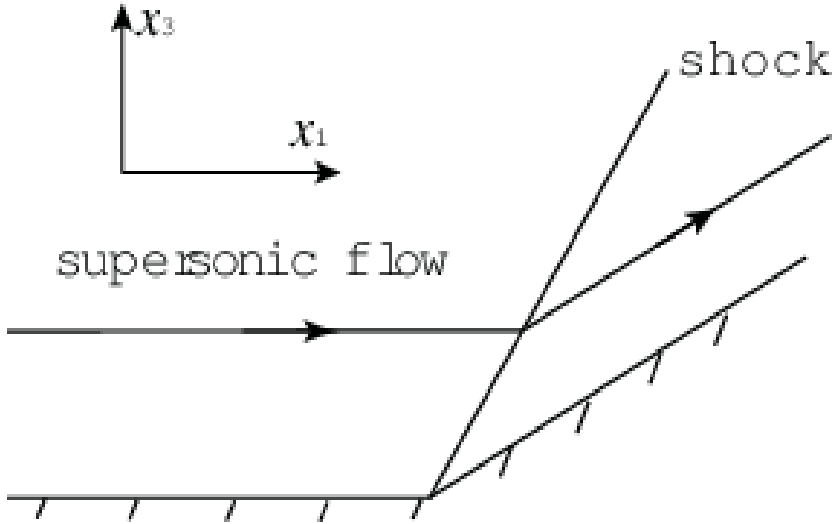


Figure 6. A uniform supersonic flow past a 2-D sharp ramp

Suppose that there is a uniform 2-D supersonic flow $(q_0, 0)$ with constant density $\rho_0 > 0$ which comes from minus infinity, and the flow hits the 2-D sharp ramp in the x_1 -direction (see the Figure 6 above). The ramp boundary is described by $x_3 = b_0 x_1$ ($b_0 > 0$), then as indicated in pages 317 of [9], there exists a critical value b^* such that there will appear a transonic shock $x_3 = s_0 x_1$ ($s_0 > b_0$) attached at the edge of ramp for $b_0 < b^*$. Moreover, it follows from Rankine-Hugoniot conditions and the boundary

condition on the ramp that the constant downstream subsonic flow $(\rho_0^+, u_{10}^+, u_{30}^+)$ satisfies

$$\begin{cases} s_0(\rho_0^+ u_{10}^+ - \rho_0 q_0) - \rho_0^+ u_{30}^+ = 0, \\ u_{10}^+ - q_0 + s_0 u_{30}^+ = 0, \\ \frac{1}{2}((u_{10}^+)^2 + (u_{30}^+)^2) + h(\rho_0^+) \equiv C_0 = \frac{q_0^2}{2} + h(\rho_0), \\ u_{30}^+ = b_0 u_{10}^+ \end{cases} \quad (2.1)$$

with

$$(u_{10}^+)^2 + (u_{30}^+)^2 < c^2(\rho_0^+). \quad (2.2)$$

In addition, the following physical entropy condition holds

$$\rho_0 < \rho_0^+. \quad (2.3)$$

With respect to the properties of the downstream subsonic flow $(\rho_0^+, u_{10}^+, u_{30}^+)$ and the slope s_0 of the transonic oblique shock, for large q_0 , we have

Lemma 2.1. *If q_0 is large and $b_0 > 0$ is fixed, then one has for $1 < \gamma < 3$*

- (i) $s_0 = \frac{1}{b_0 \rho_0} \left(\frac{\gamma-1}{2A\gamma} \right)^{\frac{1}{\gamma-1}} q_0^{\frac{2}{\gamma-1}} (1 + O(q_0^{-\frac{2}{\gamma-1}}) + O(q_0^{-2}))$.
- (ii) $u_{10}^+ = O(q_0^{\frac{\gamma-3}{\gamma-1}})$.
- (iii) $u_{30}^+ = O(q_0^{\frac{\gamma-3}{\gamma-1}})$.
- (iv) $\rho_0^+ = \left(\frac{\gamma-1}{2A\gamma} \right)^{\frac{1}{\gamma-1}} q_0^{\frac{2}{\gamma-1}} (1 + O(q_0^{-2}) + O(q_0^{-\frac{2}{\gamma-1}}))$.
- (v) $c^2(\rho_0^+) = \frac{\gamma-1}{2} q_0^2 (1 + O(q_0^{-2}) + O(q_0^{-\frac{4}{\gamma-1}}))$.
- (vi) $(q_0^+)^2 - c^2(\rho_0^+) = -\frac{\gamma-1}{2} q_0^2 (1 + O(q_0^{-2}) + O(q_0^{-\frac{4}{\gamma-1}}))$, here and below $(q_0^+)^2 = (u_{10}^+)^2 + (u_{30}^+)^2$.

Proof. (i) It follows from (2.1) that

$$\begin{cases} u_{10}^+ = q_0 - \frac{s_0^2 q_0 (\rho_0^+ - \rho_0)}{(1 + s_0^2) \rho_0^+}, \\ u_{30}^+ = \frac{s_0 q_0 (\rho_0^+ - \rho_0)}{(1 + s_0^2) \rho_0^+}, \\ h(\rho_0^+) - h(\rho_0) - \frac{s_0^2 q_0^2 ((\rho_0^+)^2 - \rho_0^2)}{2(1 + s_0^2) (\rho_0^+)^2} = 0. \end{cases} \quad (2.4)$$

From the third equation in (2.4), we have

$$\frac{A\gamma}{\gamma-1} ((\rho_0^+)^{\gamma-1} - \rho_0^{\gamma-1}) = \frac{s_0^2 q_0^2}{2(1 + s_0^2)} \left(1 - \left(\frac{\rho_0}{\rho_0^+} \right)^2 \right).$$

Denoting by $\alpha = \frac{\rho_0^+}{\rho_0}$, then one has

$$\alpha^{\gamma-1} = 1 + \frac{\rho_0^{1-\gamma} (\gamma-1) q_0^2}{2A\gamma} \left(1 - \frac{1}{1 + s_0^2} \right) \left(1 - \frac{1}{\alpha^2} \right). \quad (2.5)$$

Therefore, for large q_0 and $\alpha > 1$, one derives $\alpha = \left(\frac{\rho_0^{1-\gamma}(\gamma-1)}{2A\gamma} \right)^{\frac{1}{\gamma-1}} q_0^{\frac{2}{\gamma-1}} (1 + O(q_0^{-2}))$ and

$$\begin{cases} u_{30}^+ = \frac{s_0 q_0}{1 + s_0^2} \left(1 - \frac{1}{\alpha}\right), \\ u_{10}^+ = \frac{q_0}{1 + s_0^2} + \frac{s_0^2 q_0}{(1 + s_0^2)\alpha}. \end{cases} \quad (2.6)$$

Furthermore, by

$$u_{30}^+ = b_0 u_{10}^+, \quad (2.7)$$

we arrive at

$$s_0 = \frac{\alpha}{b_0} \left(1 - \frac{1}{\alpha} - \frac{b_0}{s_0}\right) = \frac{1}{b_0 \rho_0} \left(\frac{\gamma-1}{2A\gamma} \right)^{\frac{1}{\gamma-1}} q_0^{\frac{2}{\gamma-1}} (1 + O(q_0^{-\frac{2}{\gamma-1}}) + O(q_0^{-2})), \quad (2.8)$$

which leads to (i) of Lemma 2.1.

(ii) and (iii) come from (2.6) and (2.8) directly.

(iv)-(vi) come from the system (2.1) and (i)-(iii). \square

Next, we introduce some weighted Hölder spaces which are motivated by the Chapter 6 of [15] and [14]. These spaces are also applied in [2], [6], [26-27], [33] and so on.

Let $D \subset \mathbb{R}^3$ be an open set including the x_2 -axis, for $x, y \in D$, we define $r_x^2 = x_1^2 + x_3^2$ and $r_{x,y} = \min(r_x, r_y)$. For $m \in \mathbb{N} \cup \{0\}$, $0 < \alpha < 1$, $k, l \in \mathbb{R}$ and $u \in C_{loc}^{m,\alpha}(\bar{D} \setminus \{(0, x_2, 0) : x_2 \in \mathbb{R}\})$, we define

$$\begin{aligned} [u]_{m,0;D}^{(k,l)} &= \max \left\{ \sup_{0 < r_x < 1} \sum_{|\beta|=m} |r_x^{\max(k+m,0)} D^\beta u(x)|, \sup_{r_x > 1} \sum_{|\beta|=m} |r_x^{l+m} D^\beta u(x)| \right\}, \\ [u]_{m,\alpha;D}^{(k,l)} &= \max \left\{ \sup_{0 < r_{x,y} < 1} \sum_{|\beta|=m} r_{x,y}^{\max(k+m+\alpha,0)} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha}, \right. \\ &\quad \left. \sup_{r_{x,y} > 1} \sum_{|\beta|=m} r_{x,y}^{l+m+\alpha} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha} \right\}, \\ \|u\|_{m,0;D}^{(k,l)} &= \sum_{j=0}^m [u]_{j,0;D}^{(k,l)}, \\ \|u\|_{m,\alpha;D}^{(k,l)} &= \|u\|_{m,0;D}^{(k,l)} + [u]_{m,\alpha;D}^{(k,l)}, \end{aligned}$$

and the related function space is defined as

$$H_{m,\alpha}^{(k,l)}(D) = \{u \in C_{loc}^{m,\alpha}(\bar{D} \setminus \{(0, x_2, 0) : x_2 \in \mathbb{R}\}) : \|u\|_{m,\alpha}^{(k,l)} < +\infty\}.$$

Let $E = D \cap \{(x_1, x_2, x_3) : x_1^2 + x_3^2 < 1, x_2 \in \mathbb{R}\}$, which is a domain near the x_2 -axis.

For $m \in \mathbb{N} \cup \{0\}$, $0 < \alpha < 1$, $k \in \mathbb{R}$ and $u \in C_{loc}^{m,\alpha}(\bar{E} \setminus \{(0, x_2, 0) : x_2 \in \mathbb{R}\})$, we define

$$\begin{aligned} [u]_{m,0;E}^{(k,\star)} &= \sup_{|\beta|=m} \sum |r_x^{\max(k+m,0)} D^\beta u(x)|, \\ [u]_{m,\alpha;E}^{(k,\star)} &= \sup_{|\beta|=m} \sum r_{x,y}^{\max(k+m+\alpha,0)} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha}, \\ \|u\|_{m,0;E}^{(k,\star)} &= \sum_{j=0}^m [u]_{j,0;E}^{(k,\star)}, \\ \|u\|_{m,\alpha;E}^{(k,\star)} &= \|u\|_{m,0;E}^{(k,\star)} + [u]_{m,\alpha;E}^{(k,\star)}, \end{aligned}$$

and the related function space is defined as

$$H_{m,\alpha}^{(k,\star)}(E) = \{u \in C_{loc}^{m,\alpha}(\bar{E} \setminus \{(0, x_2, 0) : x_2 \in \mathbb{R}\}) : \|u\|_{m,\alpha}^{(k,\star)} < +\infty\}.$$

Analogously, set $F = D \cap \{(x_1, x_2, x_3) : x_1^2 + x_3^2 > 1, x_2 \in \mathbb{R}\}$ which is a domain away from x_2 -axis. We define for $m \in \mathbb{N} \cup \{0\}$, $0 < \alpha < 1$, $l \in \mathbb{R}$ and $u \in C_{loc}^{m,\alpha}(\bar{F})$,

$$\begin{aligned} [u]_{m,0;F}^{(\star,l)} &= \sup_{|\beta|=m} \sum |r_x^{l+m} D^\beta u(x)|, \\ [u]_{m,\alpha;F}^{(\star,l)} &= \sup_{|\beta|=m} \sum r_{x,y}^{l+m+\alpha} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha}, \\ \|u\|_{m,0;F}^{(\star,l)} &= \sum_{j=0}^m [u]_{j,0;F}^{(\star,l)}, \\ \|u\|_{m,\alpha;F}^{(\star,l)} &= \|u\|_{m,0;F}^{(\star,l)} + [u]_{m,\alpha;F}^{(\star,l)}, \end{aligned}$$

and the related function space is defined as

$$H_{m,\alpha}^{(\star,l)}(F) = \{u \in C_{loc}^{m,\alpha}(\bar{F}) : \|u\|_{m,\alpha}^{(\star,l)} < +\infty\}.$$

From the definitions of $H_{m,\alpha}^{(k,l)}(D)$, $H_{m,\alpha}^{(k,\star)}(E)$ and $H_{m,\alpha}^{(\star,l)}(F)$, one easily knows that the space $H_{m,\alpha}^{(k,l)}(D)$ can be split into the two subspaces $H_{m,\alpha}^{(k,\star)}(E)$ and $H_{m,\alpha}^{(\star,l)}(F)$.

For the domain E defined above, we set $E_\sigma = \{x \in E : r_x > \sigma\}$ for some positive constant $\sigma > 0$. The following weighted Hölder space $H_a^{(b)}(E)$ ($a > 0, b \in \mathbb{R}$) was introduced in [14]:

$$H_a^{(b)}(E) = \{u(x) \in C_{loc}^a(E) : \sup_{\sigma>0} \sigma^{a+b} \|u\|_{a;E_\sigma} < \infty\},$$

where $\|\cdot\|_{a;E_\sigma}$ stands for the norm of the Hölder space $C^a(E_\sigma)$. In addition, as in [14], we denote by

$$|u|_{a;E}^{(b)} = \sup_{\sigma>0} \sigma^{a+b} \|u\|_{a;E_\sigma}.$$

Then we have

Lemma 2.2. (i) If $a > 0$, then $H_a^{(-a)}(E) = C^a(E)$.

(ii) If $a \geq a' \geq 0$, $a' + b \geq 0$, $b \notin \mathbb{N}$, then $|u|_{a';E}^{(b)} \leq C|u|_{a;E}^{(b)}$.

(iii) If $0 < b < 1$ and $0 < a < 1$, then $|u|_{a;E}^{(b)} \leq C||u||_{0,a;E}^{(b,*)}$.

Proof. (i) and (ii) can be found in Lemma 2.1 of [14]. We now prove (iii). Noticing that for any $x \in E_\sigma$, one has $\sigma \leq r_x \leq 1$. This derives $\sigma^{a+b}|u(x)| \leq r_x^{a+b}|u(x)| \leq r_x^b|u(x)|$ and

$$\sup_{\sigma>0} (\sigma^{a+b} \sup_{x \in E_\sigma} |u(x)|) \leq \sup_{x \in E} r_x^b |u(x)|. \quad (2.9)$$

And similarly, we have

$$\sup_{\sigma>0} (\sigma^{a+b} \sup_{x,y \in E_\sigma} \frac{|u(x) - u(y)|}{|x - y|^a}) \leq \sup_{x,y \in E} r_{x,y}^{a+b} \frac{|u(x) - u(y)|}{|x - y|^a}. \quad (2.10)$$

Therefore it follows from (2.9)-(2.10) that (iii) holds. \square

In order to apply the Sturm-Liouville theorem and separation variable method to solve the linearized problem of (1.19), we require to list or establish some properties on the modified Bessel functions $I_\nu(t)$ and $K_\nu(t)$ of the first and second kind of order ν ($\nu \in \mathbb{R}$) respectively, where $t \in \mathbb{R}$, and $I_\nu(t)$ and $K_\nu(t)$ are two linearly independent solutions to the ordinary differential equation $t^2 \frac{d^2 w}{dt^2} + t \frac{dw}{dt} + (t^2 - \nu^2)w = 0$.

Lemma 2.3. For $I_\nu(t)$ and $K_\nu(t)$, we have

(i) $I_\nu(t)$ and $K_\nu(t)$ have the following integral representations:

$$I_\nu(t) = \frac{2e^t(2t)^\nu}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})} \int_0^1 e^{-2tu^2} u^{2\nu} (1 - u^2)^{\nu - \frac{1}{2}} du \quad \text{when } \operatorname{Re} \nu > -\frac{1}{2},$$

$$K_\nu(t) = \frac{\sqrt{\pi}e^{-t}}{\sqrt{2t}\Gamma(\nu + \frac{1}{2})} \int_0^\infty e^{-u} u^{\nu - \frac{1}{2}} (1 + \frac{u}{2t})^{\nu - \frac{1}{2}} du \quad \text{when } \operatorname{Re} \nu > -\frac{1}{2} \quad \text{and } t > 0.$$

(ii) $I'_\nu(t) = I_{\nu+1}(t) + \frac{\nu}{t} I_\nu(t)$ and $K'_\nu(t) = -K_{\nu-1}(t) - \frac{\nu}{t} K_\nu(t)$.

Especially, $I'_0(t) = I_1(t)$ and $K'_0(t) = -K_{-1}(t) = -K_1(t)$.

(iii) For any $t > 0$, then

$$(a) \quad I_0(t) \leq e^t,$$

$$(b) \quad K_0(t) \leq \frac{\sqrt{\pi}e^{-t}}{\sqrt{2t}}.$$

(iv) If $\nu > \frac{1}{2}$ and $t < 1$, then

$$(a) \quad 0 < I_\nu(t) \leq \frac{e^t(\frac{t}{2})^\nu}{\Gamma(\nu + 1)},$$

$$(b) \quad 0 < K_\nu(t) \leq \frac{e^t \Gamma(\nu) 2^{\nu-1}}{t^\nu}.$$

(v) If $\nu > \frac{1}{2}$ and $t \geq 1$, then

$$0 < I_\nu(t) \leq \frac{e^t}{\sqrt{2\pi t}}.$$

(vi) If $\frac{1}{2} < \nu \leq M$ and $t \geq 1$, then there exists a constant $C_M > 0$ independent of ν such that

$$0 < K_\nu(t) \leq C_M \frac{\sqrt{\pi} e^{-t}}{\sqrt{2t}}.$$

(vii) When $|x|$ is large and $\mu = 4\nu^2$, then the following asymptotic expansions hold

$$I'_\nu(t) \sim \frac{e^t}{\sqrt{2\pi t}} \left(1 - \frac{\mu+3}{8t} + \frac{(\mu-1)(\mu+15)}{2!(8t)^2} - \frac{(\mu-1)(\mu-9)(\mu+35)}{3!(8t)^3} + \dots \right),$$

$$K'_\nu(t) \sim -\sqrt{\frac{\pi}{2t}} e^{-t} \left(1 + \frac{\mu+3}{8t} + \frac{(\mu-1)(\mu+15)}{2!(8t)^2} + \frac{(\mu-1)(\mu-9)(\mu+35)}{3!(8t)^3} + \dots \right).$$

(viii) When ν is large, the following expansions hold uniformly with respect to t

$$I_\nu(\nu t) \sim \frac{1}{\sqrt{2\pi\nu}} \frac{e^{\nu\eta(t)}}{(1+t^2)^{\frac{1}{4}}} \left(1 + \sum_{k=1}^{\infty} \frac{u_k(\tau(t))}{\nu^k} \right),$$

$$K_\nu(\nu t) \sim \sqrt{\frac{\pi}{2\nu}} \frac{e^{-\nu\eta(t)}}{(1+t^2)^{\frac{1}{4}}} \left(1 + \sum_{k=1}^{\infty} (-1)^k \frac{u_k(\tau(t))}{\nu^k} \right),$$

$$I'_\nu(\nu t) \sim \frac{1}{\sqrt{2\pi\nu}} \frac{(1+t^2)^{\frac{1}{4}}}{t} e^{\nu\eta(t)} \left(1 + \sum_{k=1}^{\infty} \frac{v_k(\tau(t))}{\nu^k} \right),$$

$$K'_\nu(\nu t) \sim -\sqrt{\frac{\pi}{2\nu}} \frac{(1+t^2)^{\frac{1}{4}}}{t} e^{-\nu\eta(t)} \left(1 + \sum_{k=1}^{\infty} (-1)^k \frac{v_k(\tau(t))}{\nu^k} \right);$$

where $\tau(t) = \frac{1}{\sqrt{1+t^2}}$, $\eta(t) = \sqrt{1+t^2} + \ln \frac{t}{1+\sqrt{1+t^2}}$, and

$$u_{k+1}(s) = \frac{1}{2} s^2 (1-s^2) u'_k(s) + \frac{1}{8} \int_0^s (1-5s^2) u_k(s) ds, \quad k = 0, 1, \dots,$$

$$v_k(s) = u_k(s) + s(s^2-1) \left(\frac{1}{2} u_{k-1}(s) + s u'_{k-1}(s) \right), \quad k = 1, 2, \dots,$$

$$u_0(s) = 1.$$

(ix) For $t_1 \leq t_2$, then $e^{\nu\eta(t_1)} e^{-\nu\eta(t_2)} \leq e^{-\nu(t_2-t_1)}$, where $\eta(t)$ has been defined in (viii).

Proof. (i)-(ii) can be found in Pages 204-206 and Pages 79 of [24], and (vii)-(viii) can be seen from Pages 377-378 of [1].

We now show (iii). It follows from (i) that for $t > 0$

$$\begin{aligned} I_0(t) &= \frac{e^t}{\sqrt{\pi}\Gamma(\frac{1}{2})} \int_0^1 e^{-2ts} (1-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds \\ &\leq \frac{e^t}{\sqrt{\pi}\Gamma(\frac{1}{2})} \int_0^1 (1-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds \\ &= e^t \end{aligned}$$

and

$$\begin{aligned} K_0(t) &\leq \frac{e^{-t}}{\sqrt{2t}} \int_0^\infty e^{-u} u^{-\frac{1}{2}} du \\ &= \frac{\sqrt{\pi} e^{-t}}{\sqrt{2t}}. \end{aligned}$$

Thus, (iii) is proved.

Next, we start to prove (iv). Since $\nu > \frac{1}{2}$ and $t < 1$, we have that from (i)

$$\begin{aligned} I_\nu(t) &\leq \frac{(2t)^\nu e^t}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_0^1 u^{\nu-\frac{1}{2}} (1-u)^{\nu-\frac{1}{2}} du \\ &= \frac{(2t)^\nu e^t}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} B(\nu + \frac{1}{2}, \nu + \frac{1}{2}) \\ &= \frac{e^t (\frac{t}{2})^\nu}{\Gamma(\nu + 1)}. \end{aligned}$$

Similarly, we have that from (i)

$$\begin{aligned} K_\nu(t) &= \frac{\sqrt{\pi} e^{-t}}{(2t)^\nu \Gamma(\nu + \frac{1}{2})} \int_0^\infty e^{-u} u^{\nu-\frac{1}{2}} (2t+u)^{\nu-\frac{1}{2}} du \\ &\leq \frac{\sqrt{\pi} e^{-t}}{(2t)^\nu \Gamma(\nu + \frac{1}{2})} \int_0^\infty e^{-u} (2t+u)^{2\nu-1} du \\ &\leq \frac{\sqrt{\pi} e^t}{(2t)^\nu} \frac{\Gamma(2\nu)}{\Gamma(\nu + \frac{1}{2})} \\ &= \frac{e^t \Gamma(\nu) 2^{\nu-1}}{t^\nu}. \end{aligned}$$

Thus, we complete the proof of (iv).

Next, we prove (v). Since $t \geq 1$ and $\nu > \frac{1}{2}$, then by (i)

$$\begin{aligned} I_\nu(t) &= \frac{e^t}{\sqrt{2\pi t} \Gamma(\nu + \frac{1}{2})} \int_0^{2t} e^{-s} s^{\nu-\frac{1}{2}} (1 - \frac{s}{2t})^{\nu-\frac{1}{2}} ds \\ &\leq \frac{e^t}{\sqrt{2\pi t} \Gamma(\nu + \frac{1}{2})} \int_0^{2t} e^{-s} s^{\nu-\frac{1}{2}} ds \\ &\leq \frac{e^t}{\sqrt{2\pi t} \Gamma(\nu + \frac{1}{2})} \Gamma(\nu + \frac{1}{2}) \\ &= \frac{e^t}{\sqrt{2\pi t}}. \end{aligned}$$

We now show (vi). Due to $\frac{1}{2} < \nu \leq M$ and $t \geq 1$, we have from (i)

$$\begin{aligned}
K_\nu(t) &= \frac{\sqrt{\pi}e^{-t}}{\sqrt{2t}\Gamma(\nu + \frac{1}{2})} \left(\int_0^{2t} e^{-u} u^{\nu-\frac{1}{2}} \left(1 + \frac{u}{2t}\right)^{\nu-\frac{1}{2}} du + \int_{2t}^\infty e^{-u} u^{\nu-\frac{1}{2}} \left(1 + \frac{u}{2t}\right)^{\nu-\frac{1}{2}} du \right) \\
&\leq \frac{\sqrt{\pi}e^{-t}}{\sqrt{2t}\Gamma(\nu + \frac{1}{2})} \left(2^{\nu-\frac{1}{2}} \int_0^{2t} e^{-u} u^{\nu-\frac{1}{2}} du + (2t)^{\frac{1}{2}-\nu} \int_{2t}^\infty e^{-u} u^{\nu-\frac{1}{2}} (2t+u)^{\nu-\frac{1}{2}} du \right) \\
&\leq \frac{\sqrt{\pi}e^{-t}}{\sqrt{2t}\Gamma(\nu + \frac{1}{2})} \left(2^{\nu-\frac{1}{2}} \Gamma(\nu + \frac{1}{2}) + t^{\frac{1}{2}-\nu} \Gamma(2\nu) \right) \\
&\leq C_M \frac{\sqrt{\pi}e^{-t}}{\sqrt{2t}}.
\end{aligned}$$

Finally, we prove (ix). By $\eta(t) = \sqrt{1+t^2} + \ln \frac{t}{1+\sqrt{1+t^2}}$, then one has $\eta'(t) = \frac{\sqrt{1+t^2}}{t}$. Thus, there exist a number $\xi \in (t_1, t_2)$ such that

$$e^{\nu\eta(t_1)} e^{-\nu\eta(t_2)} = e^{-\nu \frac{\sqrt{1+\xi^2}}{\xi} (t_2-t_1)} \leq e^{-\nu(t_2-t_1)}.$$

Collecting all the analysis above, we complete the proof of Lemma 2.3. \square

Finally, we give an existence result of the supersonic solution to (1.5) and (1.13) in the domain which is larger than that of left hand side of the shock surface Γ .

Lemma 2.4. *The equation (1.5) with the initial data (1.13) has a C^∞ solution $\varphi^-(x)$ in the domain $\Omega_- = \{x : x_1 \geq 0, x_2 \in \mathbb{R}, x_3 \geq \frac{s_0 + b_0}{2} x_1\}$. Moreover, $\varphi^-(x) - q_0 x_1 \in C^\infty(\Omega_-)$, $\varphi^-(x) = \varphi^-(x_1, x_2 + 2\pi, x_3)$, and there exists a positive constant C_k independent of ε such that*

$$\|\varphi^-(x) - q_0 x_1\|_{C^k(\Omega_-)} \leq C_k \varepsilon \quad (2.11)$$

for any fixed $k \in \mathbb{N}$.

Proof. We note that the equation (1.5) is quasi-linear strictly hyperbolic with respect to the x_1 -direction for the supersonic flow $\partial_1 \varphi^- > c^-$, furthermore, the initial condition (1.13) is of a small perturbation. Thus, in terms of the finite propagation property of the wave equation, the periodic property of the initial data $(\varphi_0(x_2, x_3), \varphi_1(x_2, x_3))$ with respect to the variable x_2 and the Picard iteration (or one can see [16]), we know that Lemma 2.4 holds. \square

Remark 2.1. *By (2.11) and the standard extension theorem (see Theorem 7.25 of [15]), we can extend the smooth function $\varphi^-(x)$ in Ω_- into the whole domain $\Omega = \{x : x_1 \geq 0, x_2 \in \mathbb{R}, x_3 \geq b_0 x_1\}$ such that the extension function $\tilde{\varphi}^-(x) \in C^\infty(\bar{\Omega})$ satisfies $\tilde{\varphi}^-(x) = \varphi^-(x)$ for $x \in \Omega_-$ and $\|\tilde{\varphi}^-(x) - q_0 x_1\|_{C^k(\bar{\Omega})} \leq C_k \varepsilon$. For convenience, $\tilde{\varphi}^-(x)$ will still be denoted by $\varphi^-(x)$ later. Here one should notice that $\tilde{\varphi}^-(x)$ is not a solution to (1.5) in $\Omega \setminus \Omega_-$ in general case.*

§3. Reformulation on (1.6)-(1.12) and detailed descriptions on Theorem 1.1

By the notations in (1.14)-(1.16) of §1, it follows from Lemma 2.1 that the function $\Phi(x)$ corresponding to the background solution is

$$\Phi_0(x) = (q_0 - u_{10}^+)x_1 - u_{30}^+x_3 = q_0x_1 + O(q_0^{\frac{\gamma-3}{\gamma-1}})(x_1 + x_3). \quad (3.1)$$

In this case, $\partial_{x_1}\Phi_0(x) = q_0 + O(q_0^{\frac{\gamma-3}{\gamma-1}}) > 0$ and $\partial_{x_3}\Phi_0(x) = O(q_0^{\frac{\gamma-3}{\gamma-1}})$ holds for large q_0 and $1 < \gamma < 3$. Thus, the transformation (1.15) is inverse since $\partial_{x_1}\Phi(x)$ and $\partial_{x_3}\Phi(x)$ will be of the small perturbations of $\partial_{x_1}\Phi_0(x)$ and $\partial_{x_3}\Phi_0(x)$ respectively. In addition, the corresponding unknown function $u(y)$ in (1.16) for $\Phi_0(x)$ can be expressed as

$$u_0(y) = y_1 + O(q_0^{-\frac{2}{\gamma-1}})(y_1 + y_3). \quad (3.2)$$

To solve (1.5)-(1.6) together with (1.7)-(1.13), we suffice to study the following problem (one can also see (1.18) in §1)

$$\left\{ \begin{array}{ll} L(u, \nabla_y u, \nabla_y^2 u) \\ \equiv \sum_{1 \leq i \leq j \leq 3} A_{ij}(u, \nabla_y u) \partial_{y_i y_j}^2 u + \frac{1}{q_0} \sum_{1 \leq i \leq j \leq 3} a_{ij}(\nabla_x \varphi^- - \nabla_x \Phi) \partial_{x_i x_j}^2 \varphi^- = 0 & \text{in } Q, \\ G_1(u, \nabla_y u) = 0 & \text{on } y_3 = b_0 y_1, \\ G_2(u, \nabla_y u) = 0 & \text{on } y_1 = 0, \\ u(0, y_2, 0) = 0, \\ u(y_1, y_2 + 2\pi, y_3) = u(y), \\ \lim_{y_1 + y_3 \rightarrow +\infty} \nabla_y u \text{ exists,} \end{array} \right. \quad (3.3)$$

where

$$\begin{aligned} G_1(u, \nabla_y u) &\equiv -\left(1 + b_0^2 + \frac{b_0}{q_0}(\partial_{x_3} \varphi^- - b_0 \partial_{x_1} \varphi^-)\right) \partial_{y_3} u - \frac{1}{q_0}(\partial_{x_3} \varphi^- - b_0 \partial_{x_1} \varphi^-) \partial_{y_1} u - b_0, \\ G_2(u, \nabla_y u) &\equiv \frac{1}{q_0} \left(1 - \frac{\rho_-}{\rho_+}\right) \partial_{x_1} \varphi^- \partial_{y_1} u + b_0 \left(\frac{1}{q_0} \left(1 - \frac{\rho_-}{\rho_+}\right) \partial_{x_1} \varphi^- - 2\right) \partial_{y_3} u \\ &\quad + \frac{b_0}{q_0} \left(1 - \frac{\rho_-}{\rho_+}\right) \partial_{x_1} \varphi^- \partial_{y_1} u \partial_{y_3} u - (\partial_{y_2} u)^2 + \left(\frac{b_0}{q_0} \left(1 - \frac{\rho_-}{\rho_+}\right) (b_0 \partial_{x_1} \varphi^- - \partial_{x_3} \varphi^-)\right. \\ &\quad \left. - (1 + b_0^2)\right) (\partial_{y_3} u)^2 - \frac{1}{q_0} \left(1 - \frac{\rho_-}{\rho_+}\right) \partial_{x_2} \varphi^- \partial_{y_1} u \partial_{y_2} u - \frac{b_0}{q_0} \left(1 - \frac{\rho_-}{\rho_+}\right) \partial_{x_2} \varphi^- \partial_{y_2} u \partial_{y_3} u \\ &\quad - \frac{1}{q_0} \left(1 - \frac{\rho_-}{\rho_+}\right) \partial_{x_3} \varphi^- \partial_{y_1} u \partial_{y_3} u - 1 \end{aligned}$$

and

$$\begin{aligned}
A_{11} &= \frac{1}{(\partial_{y_1} u + b_0 \partial_{y_3} u)^3} \left(a_{11}(1 + b_0 \partial_{y_3} u)^2 + a_{22}(\partial_{y_2} u)^2 + a_{33}(\partial_{y_3} u)^2 - 2a_{12} \partial_{y_2} u(1 + b_0 \partial_{y_3} u) \right. \\
&\quad \left. - 2a_{13} \partial_{y_3} u(1 + b_0 \partial_{y_3} u) + 2a_{23} \partial_{y_2} u \partial_{y_3} u \right), \\
A_{22} &= \frac{a_{22}}{\partial_{y_1} u + b_0 \partial_{y_3} u}, \\
A_{33} &= \frac{1}{(\partial_{y_1} u + b_0 \partial_{y_3} u)^3} \left(a_{11} b_0^2 (1 - \partial_{y_1} u)^2 + a_{22} b_0^2 (\partial_{y_2} u)^2 + a_{33} (\partial_{y_1} u)^2 - 2a_{12} b_0^2 \partial_{y_2} u(1 - \partial_{y_1} u) \right. \\
&\quad \left. + 2a_{13} b_0 \partial_{y_1} u(1 - \partial_{y_1} u) - 2a_{23} b_0 \partial_{y_1} u \partial_{y_2} u \right), \\
A_{12} = A_{21} &= \frac{1}{(\partial_{y_1} u + b_0 \partial_{y_3} u)^2} \left(-a_{22} \partial_{y_2} u + a_{12}(1 + b_0 \partial_{y_3} u) - a_{23} \partial_{y_3} u \right), \\
A_{13} = A_{31} &= \frac{1}{(\partial_{y_1} u + b_0 \partial_{y_3} u)^3} \left(a_{11} b_0 (1 + b_0 \partial_{y_3} u)(1 - \partial_{y_1} u) + a_{22} b_0 (\partial_{y_2} u)^2 - a_{33} \partial_{y_1} u \partial_{y_3} u \right. \\
&\quad \left. + a_{12} b_0 \partial_{y_2} u (\partial_{y_1} u - b_0 \partial_{y_3} u - 2) + a_{13} (\partial_{y_1} u + 2b_0 \partial_{y_1} u \partial_{y_3} u - b_0 \partial_{y_3} u) \right. \\
&\quad \left. - a_{23} \partial_{y_2} u (\partial_{y_1} u + b_0 \partial_{y_3} u) \right), \\
A_{23} = A_{32} &= \frac{1}{(\partial_{y_1} u + b_0 \partial_{y_3} u)^2} \left(-a_{22} b_0 \partial_{y_2} u + a_{12} b_0 (1 - \partial_{y_1} u) + a_{23} \partial_{y_1} u \right).
\end{aligned}$$

With respect to more precise properties of $G_1(u, \nabla_y u)$, $G_2(u, \nabla_y u)$ and A_{ij} , one can be referred in §7 below. In addition, one should note that $\varphi^-(x)$ in (3.3) has become a function $\varphi^-(u(y), y_2, y_3 - b_0 y_1 + b_0 u(y))$ depending on the unknown solution $u(y)$, and $\nabla_x \Phi(x)$ in (3.3) is also a function on $\nabla_y u$ by the transformations (1.15)-(1.16). On the other hand, we especially point out that the condition $u(0, y_2, 0) \equiv 0$ for all $y_2 \in \mathbb{R}$ in (3.3) comes from the attached shock property. Next we will show that (3.3) is overdetermined since (3.3) can be solved as long as the condition $u(0, y_2, 0) \equiv 0$ in (3.3) is replaced by $u(0, y_2^0, 0) = 0$ for any fixed y_2^0 . Without loss of generality, we assume $y_2^0 = 0$ and consider the following problem instead of (3.3)

$$\begin{cases}
L(u, \nabla_y u, \nabla_y^2 u) = 0 & \text{in } Q, \\
G_1(u, \nabla_y u) = 0 & \text{on } y_3 = b_0 y_1, \\
G_2(u, \nabla_y u) = 0 & \text{on } y_1 = 0, \\
u(0, 0, 0) = 0, \\
u(y_1, y_2 + 2\pi, y_3) = u(y), \\
\lim_{y_1 + y_3 \rightarrow +\infty} \nabla_y u \text{ exists.}
\end{cases} \quad (3.4)$$

With respect to the problem (3.4), we have

Theorem 3.1. *There exist some positive constants $\varepsilon_0 > 0$, $0 < \delta < 1$, $0 < \delta_0 < 1$, and $C > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, the problem (3.4) has a unique solution $u \in C^{6,\alpha}(Q)$*

which fulfills the following estimate

$$\|u - u_0\|_{6,\alpha;Q}^{(-1-\delta,-\delta_0)} \leq C\varepsilon,$$

where $0 < \alpha < 1$.

Remark 3.1. By Theorem 3.1, we have obtained the $C^{6,\alpha}$ -regularities of u in the domain Q due to the high regularity assumption on the supersonic incoming flow. On the other hand, as in [27], by use of the separation variable method in §4 below, it seems that the $C^{6,\alpha}$ -regularities of u in the interior of Q are required to guarantee the convergence of the appropriate solution sequence in the C_{loc}^2 -space for the boundary value problem of the Laplacian equation with two Neumann boundary conditions and a vanishing condition of the first order derivatives at infinity. However, such high regularities ($C^{6,\alpha}(Q)$ -regularities) are essentially unnecessary (only $C^{2,\alpha}(Q)$ -regularities should be enough) if we establish the L_{loc}^2 -convergence of the appropriate solution sequence instead of the C_{loc}^2 convergence in any compact subdomain of Q and combine with some interior estimate techniques on the second order linear elliptic equations in Chapter 6 of [15]. Since the requirements on the higher order interior regularities are not essential for our problem, we omit the related argument procedure on the reduction from $C^{6,\alpha}(Q)$ -regularities to $C^{2,\alpha}(Q)$ -regularities in Theorem 3.1.

§4. On the linearization of (3.4) and its related cut-off problem

In order to solve the nonlinear problem (3.4), we first consider its linearized case, which corresponds to an Neumann boundary problem of a second order elliptic equation in an unbounded angular domain. It will be seen that in terms of the smallness of $\frac{1}{q_0}$ and Lemma 2.1, by a tedious but direct computation (see §7 below), the linearized problem of (3.4) can be essentially expressed as

$$\left\{ \begin{array}{ll} \Delta \dot{u} = \dot{f} & \text{in } Q, \\ \frac{\partial \dot{u}}{\partial n} = \dot{g}_2 & \text{on } \Sigma_1 : y_3 = b_0 y_1, \\ \frac{\partial \dot{u}}{\partial n} = \dot{g}_1 & \text{on } \Sigma_2 : y_1 = 0, \\ \dot{u}(y_1, y_2 + 2\pi, y_3) = \dot{u}(y), \\ \dot{u}(0, 0, 0) = 0, \\ \lim_{y_1+y_3 \rightarrow \infty} \nabla \dot{u} = 0, \end{array} \right. \quad (4.1)$$

where $\dot{f} \in H_{4,\alpha}^{(1-\delta,2-\delta_0)}(Q)$ and $\dot{g}_i \in H_{5,\alpha}^{(-\delta,1-\delta_0)}(Q)$ for $0 < \alpha < 1$ and $i = 1, 2$.

Introducing the following cylindrical coordinate transformation

$$y_1 = r \cos \theta, \quad y_2 = y_2, \quad y_3 = r \sin \theta,$$

where $r = \sqrt{y_1^2 + y_3^2}$, $\theta \in [\theta_0, \frac{\pi}{2}]$, and $\theta_0 = \arctan b_0$. Then (4.1) can be changed as

$$\begin{cases} \Delta \dot{u} = \partial_r^2 \dot{u} + r^{-2} \partial_\theta^2 \dot{u} + \partial_{y_2}^2 \dot{u} + r^{-1} \partial_r \dot{u} = \dot{f} & \text{in } Q, \\ -\partial_\theta \dot{u} = r \dot{g}_1 & \text{on } \Sigma_1 : \theta = \theta_0, \\ \partial_\theta \dot{u} = r \dot{g}_2 & \text{on } \Sigma_2 : \theta = \frac{\pi}{2}, \\ \dot{u}(r, \theta, y_2 + 2\pi) = \dot{u}(r, \theta, y_2), \\ \dot{u}(0, \theta, 0) = 0, \\ \lim_{r \rightarrow \infty} \nabla \dot{u} = 0, \end{cases} \quad (4.2)$$

where $Q = \{(r, \theta, y_2) : r \in \mathbb{R}^+, \theta \in (\theta_0, \frac{\pi}{2}), y_2 \in \mathbb{T}\}$ under the cylindrical coordinate transformation.

Let $h(r, \theta, y_2) = -\frac{r}{2(\frac{\pi}{2} - \theta_0)}(\dot{g}_1 + \dot{g}_2)(\theta - \theta_0)^2 + r \dot{g}_1 \theta$ and $v(r, \theta, y_2) = \dot{u}(r, \theta, y_2) - h(r, \theta, y_2)$, then the problem (4.2) can be changed as

$$\begin{cases} \Delta v = \partial_r^2 v + r^{-2} \partial_\theta^2 v + \partial_{y_2}^2 v + r^{-1} \partial_r v = f \equiv \dot{f} - \Delta h & \text{in } Q, \\ \partial_\theta v = 0 & \text{on } \Sigma_1, \\ \partial_\theta v = 0 & \text{on } \Sigma_2, \\ v(r, \theta, y_2 + 2\pi) = v(r, \theta, y_2), \\ v(0, \theta, 0) = 0, \\ \lim_{r \rightarrow \infty} \nabla v = 0, \end{cases} \quad (4.3)$$

where $f \in H_{4,\alpha}^{(1-\delta, 2-\delta_0)}(Q)$.

In order to solve the unbounded domain problem (4.3), we will consider the following cut-off problem in the bounded domain:

$$\begin{cases} \Delta v_L = \partial_r^2 v_L + r^{-2} \partial_\theta^2 v_L + \partial_{y_2}^2 v_L + r^{-1} \partial_r v_L = f_L \\ \quad \text{in } Q_L = \{(r, \theta, y_2) : 0 < r < L, (r, \theta, y_2) \in Q\}, \\ \partial_\theta v_L = 0 & \text{on } \Sigma_{1,L} = \{(r, y_2) : 0 < r < L, (r, y_2) \in \Sigma_1\}, \\ \partial_\theta v_L = 0 & \text{on } \Sigma_{2,L} = \{(r, y_2) : 0 < r < L, (r, y_2) \in \Sigma_2\}, \\ \partial_r v_L = c_L & \text{on } \Sigma_{3,L} = \{(r, \theta, y_2) : r = L, \theta_0 \leq \theta \leq \frac{\pi}{2}, y_2 \in \mathbb{T}\}, \\ v_L(r, \theta, y_2 + 2\pi) = v_L(r, \theta, y_2), \\ v_L(0, \theta, 0) = 0, \end{cases} \quad (4.4)$$

where $L \geq 4$, f_L is the restriction of f on Q_L , which obviously obeys

$$\|f_L\|_{4,\alpha;Q_L}^{(1-\delta, 2-\delta_0)} \leq \|f\|_{4,\alpha;Q}^{(1-\delta, 2-\delta_0)}. \quad (4.5)$$

In addition, in order to guarantee the solvability of (4.4), we require to choose the constant c_L in (4.4) such that

$$\int_{Q_L} f_L dy = \int_{\Sigma_{3,L}} c_L dS. \quad (4.6)$$

From (4.6), we can arrive at

$$|c_L| \leq CL^{\delta_0-1} \|f_L\|_{4,\alpha;Q_L}^{(1-\delta,2-\delta_0)} \leq CL^{\delta_0-1} \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)} \rightarrow 0 \quad \text{as } L \rightarrow \infty. \quad (4.7)$$

With respect to the linear problem (4.4), we have

Proposition 4.1. *There exists a unique solution $v_L \in C^2(\bar{Q}_L \setminus \{(0, y_2, 0) : y_2 \in \mathbb{R}\})$ to (4.4) such that*

$$\|v_L\|_{0,0;Q_L}^{(-1-\delta,-\delta_0)} \leq C \|f_L\|_{4,\alpha;Q_L}^{(1-\delta,2-\delta_0)}, \quad (4.8)$$

where the constant $C > 0$ is independent of L .

Proof. We will divide the proof of Proposition 4.1 into the following three steps.

Step 1. Existence of a formal solution v_L to (4.4)

We will use the separation variable method to solve (4.4). To this end, as in [29], we first focus on the corresponding homogeneous equation of (4.4). Consider the nontrivial solutions to the following problem

$$\begin{cases} \Delta v = \partial_r^2 v + r^{-2} \partial_\theta^2 v + \partial_{y_2}^2 v + r^{-1} \partial_r v = 0, \\ \partial_\theta v = 0 \quad \text{on} \quad \Sigma_{1,L} \cup \Sigma_{2,L}, \\ v(r, \theta, y_2) = v(r, \theta, y_2 + 2\pi). \end{cases} \quad (4.9)$$

Set $v(r, \theta, y_2) = R(r)\Theta(\theta)Y(y_2)$, then we have

$$\begin{cases} Y''(y_2) + \lambda Y(y_2) = 0, \\ Y(y_2) = Y(y_2 + 2\pi), \end{cases} \quad (4.10)$$

and

$$\begin{cases} \Theta''(\theta) + \mu \Theta(\theta) = 0, \\ \Theta'(\theta_0) = \Theta'(\frac{\pi}{2}) = 0, \end{cases} \quad (4.11)$$

and

$$\begin{cases} r^2 R''(r) + r R'(r) - (\mu + \lambda r^2) R(r) = 0, \\ R'(L) = 0, \quad R(0) \text{ is bounded,} \end{cases}$$

where $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$.

We can get that the eigenvalues of (4.10) and (4.11) are $\lambda_n = n^2 (n = 0, 1, \dots)$ and $\mu_m = (\frac{m\pi}{\frac{\pi}{2} - \theta_0})^2 (m = 0, 1, \dots)$, whose corresponding eigenfunctions are $\{\sin(ny_2), \cos(ny_2)\}_{n=0}^\infty$ and $\{\cos \sqrt{\mu_m}(\theta - \theta_0)\}_{m=0}^\infty$ respectively.

We now solve equation (4.4) by use of the eigenfunction expansion method in terms of the complete orthogonal basis $\{\cos \sqrt{\mu_m}(\theta - \theta_0) \sin(ny_2), \cos \sqrt{\mu_m}(\theta - \theta_0) \cos(ny_2)\}_{m,n=0}^\infty$.

Let

$$\begin{aligned}
v_L(r, \theta, y_2) &= R_{00}(r) + \sum_{m=1}^{\infty} R_{m0}(r) \cos \sqrt{\mu_m}(\theta - \theta_0) + \sum_{n=1}^{\infty} \left(R_{0n}^{(1)}(r) \sin ny_2 + R_{0n}^{(2)}(r) \cos ny_2 \right) \\
&+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(R_{mn}^{(1)}(r) \cos \sqrt{\mu_m}(\theta - \theta_0) \sin ny_2 + R_{mn}^{(2)}(r) \cos \sqrt{\mu_m}(\theta - \theta_0) \cos ny_2 \right)
\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}
f_L(r, \theta, y_2) &= f_{L00}(r) + \sum_{m=1}^{\infty} f_{Lm0}(r) \cos \sqrt{\mu_m}(\theta - \theta_0) + \sum_{n=1}^{\infty} \left(f_{L0n}^{(1)}(r) \sin ny_2 + f_{L0n}^{(2)}(r) \cos ny_2 \right) \\
&+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(f_{Lmn}^{(1)}(r) \cos \sqrt{\mu_m}(\theta - \theta_0) \sin ny_2 + f_{Lmn}^{(2)}(r) \cos \sqrt{\mu_m}(\theta - \theta_0) \cos ny_2 \right),
\end{aligned} \tag{4.13}$$

where

$$\begin{aligned}
f_{L00}(r) &= \frac{1}{2\pi(\frac{\pi}{2} - \theta_0)} \int_{\theta_0}^{\frac{\pi}{2}} \int_0^{2\pi} f_L(r, \theta, y_2) dy_2 d\theta, \\
f_{Lm0}(r) &= \frac{1}{\pi(\frac{\pi}{2} - \theta_0)} \int_{\theta_0}^{\frac{\pi}{2}} \int_0^{2\pi} f_L(r, \theta, y_2) \cos \sqrt{\mu_m}(\theta - \theta_0) dy_2 d\theta, \\
f_{L0n}^{(1)}(r) &= \frac{1}{\pi(\frac{\pi}{2} - \theta_0)} \int_{\theta_0}^{\frac{\pi}{2}} \int_0^{2\pi} f_L(r, \theta, y_2) \sin ny_2 dy_2 d\theta, \\
f_{L0n}^{(2)}(r) &= \frac{1}{\pi(\frac{\pi}{2} - \theta_0)} \int_{\theta_0}^{\frac{\pi}{2}} \int_0^{2\pi} f_L(r, \theta, y_2) \cos ny_2 dy_2 d\theta, \\
f_{Lmn}^{(1)}(r) &= \frac{2}{\pi(\frac{\pi}{2} - \theta_0)} \int_{\theta_0}^{\frac{\pi}{2}} \int_0^{2\pi} f_L(r, \theta, y_2) \cos \sqrt{\mu_m}(\theta - \theta_0) \sin ny_2 dy_2 d\theta, \\
f_{Lmn}^{(2)}(r) &= \frac{2}{\pi(\frac{\pi}{2} - \theta_0)} \int_{\theta_0}^{\frac{\pi}{2}} \int_0^{2\pi} f_L(r, \theta, y_2) \cos \sqrt{\mu_m}(\theta - \theta_0) \cos ny_2 dy_2 d\theta.
\end{aligned}$$

Substituting (4.12) and (4.13) into the equation $\Delta v_L = f_L$ yields

$$\begin{cases} R_{00}''(r) + r^{-1}R_{00}'(r) = f_{L00}(r), \\ R_{m0}''(r) - r^{-2}\mu_m R_{m0}(r) + r^{-1}R_{m0}'(r) = f_{Lm0}(r), & m \geq 1, \\ (R_{0n}^{(i)})''(r) - n^2(R_{0n}^{(i)})(r) + r^{-1}(R_{0n}^{(i)})'(r) = f_{L0n}^{(i)}(r), & n \geq 1, \quad i = 1, 2, \\ (R_{mn}^{(i)})''(r) - (r^{-2}\mu_m + n^2)(R_{mn}^{(i)})(r) + r^{-1}(R_{mn}^{(i)})'(r) = f_{Lmn}^{(i)}(r), & m, n \geq 1, \quad i = 1, 2. \end{cases} \tag{4.14}$$

Meanwhile, substituting (4.12) into the condition $v_L(0, \theta, 0) = 0$ yields

$$\left(R_{00}(0) + \sum_{n=1}^{\infty} R_{0n}^{(2)}(0) \right) + \sum_{m=1}^{\infty} \left(R_{m0}(0) + \sum_{n=1}^{\infty} R_{mn}^{(2)}(0) \right) \cos \sqrt{\mu_m}(\theta - \theta_0) = 0.$$

By the orthogonality of $\{\cos \sqrt{\mu_m}(\theta - \theta_0)\}_{m=0}^{\infty}$, then

$$R_{m0}(0) + \sum_{n=1}^{\infty} R_{mn}^{(2)}(0) = 0, \quad m = 0, 1, 2, \dots \quad (4.15)$$

In addition, it follows from $\partial_r v_L = c_L$ on $\Sigma_{3,L}$ that

$$\begin{aligned} & R'_{00}(L) + \sum_{m=1}^{\infty} R'_{m0}(L) \cos \sqrt{\mu_m}(\theta - \theta_0) + \sum_{n=1}^{\infty} \left((R_{0n}^{(1)})'(L) \sin ny_2 + (R_{0n}^{(2)})'(L) \cos ny_2 \right) \\ & + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left((R_{mn}^{(1)})'(L) \cos \sqrt{\mu_m}(\theta - \theta_0) \sin ny_2 + (R_{mn}^{(2)})'(L) \cos \sqrt{\mu_m}(\theta - \theta_0) \cos ny_2 \right) \\ & = c_L. \end{aligned}$$

This derives

$$\begin{cases} R'_{00}(L) = c_L, & R'_{m0}(L) = 0, \quad m = 1, 2, \dots, \\ (R_{mn}^{(1)})'(L) = (R_{mn}^{(2)})'(L) = 0, & m = 0, 1, 2, \dots, n = 1, 2, \dots \end{cases} \quad (4.16)$$

Collecting (4.14)-(4.16), we obtain the following systems

$$\begin{cases} R''_{00}(r) + r^{-1}R'_{00}(r) = f_{L00}(r), \\ R'_{00}(L) = c_L, \\ R_{00}(0) = - \sum_{n=1}^{\infty} R_{0n}^{(2)}(0), \end{cases} \quad (4.17)$$

$$\begin{cases} R''_{m0}(r) - r^{-2}\mu_m R_{m0}(r) + r^{-1}R'_{m0}(r) = f_{Lm0}(r), & m = 1, 2, \dots, \\ R'_{m0}(L) = 0, \\ R_{m0}(0) = - \sum_{n=1}^{\infty} R_{mn}^{(2)}(0), \\ R_{m0}(0) \text{ is bounded}, \end{cases} \quad (4.18)$$

$$\begin{cases} (R_{0n}^{(i)})''(r) - n^2 R_{0n}^{(i)}(r) + r^{-1}(R_{0n}^{(i)})'(r) = f_{L0n}^{(i)}(r), & n = 1, 2, \dots, \\ (R_{0n}^{(i)})'(L) = 0, \\ R_{0n}^{(i)}(0) \text{ is bounded}, \end{cases} \quad (4.19)$$

and

$$\begin{cases} (R_{mn}^{(i)})''(r) - (r^{-2}\mu_m + n^2)R_{mn}^{(i)}(r) + r^{-1}(R_{mn}^{(i)})'(r) = f_{Lmn}^{(i)}(r), & m, n = 1, 2, \dots, \\ (R_{mn}^{(i)})''(L) = 0, \\ R_{mn}^{(i)}(0) \text{ is bounded.} \end{cases} \quad (4.20)$$

Notice that the general solutions to the equations in (4.17)-(4.20) are respectively

$$\begin{cases} R_{00}(r) = C_{00}^1 + C_{00}^2 \ln r + \int_0^r \eta^{-1} \int_0^\eta \xi f_{L00}(\xi) d\xi d\eta, \\ R_{m0}(r) = C_m^1 r^{\sqrt{\mu_m}} + C_m^2 r^{-\sqrt{\mu_m}} + r^{-\sqrt{\mu_m}} \int_r^L \eta^{2\sqrt{\mu_m}-1} \int_\eta^L \xi^{1-\sqrt{\mu_m}} f_{Lm0}(\xi) d\xi d\eta, \\ \hspace{15em} m = 1, 2, \dots, \\ R_{0n}^{(i)}(r) = C_{0n}^{i1} I_0(nr) + C_{0n}^{i2} K_0(nr) - I_0(nr) \int_r^L s K_0(ns) f_{L0n}^{(i)}(s) ds \\ \hspace{4em} - K_0(nr) \int_0^r s I_0(ns) f_{L0n}^{(i)}(s) ds, \quad i = 1, 2, \quad n = 1, 2, \dots, \\ R_{mn}^{(i)}(r) = C_{mn}^{i1} I_{\sqrt{\mu_m}}(nr) + C_{mn}^{i2} K_{\sqrt{\mu_m}}(nr) - I_{\sqrt{\mu_m}}(nr) \int_r^L s K_{\sqrt{\mu_m}}(ns) f_{Lmn}^{(i)}(s) ds \\ \hspace{4em} - K_{\sqrt{\mu_m}}(nr) \int_0^r s I_{\sqrt{\mu_m}}(ns) f_{Lmn}^{(i)}(s) ds, \quad i = 1, 2, \quad m, n = 1, 2, \dots, \end{cases} \quad (4.21)$$

Next, we determine the constants in (4.21) by the boundary conditions in (4.17)-(4.20). At first, due to $R'_{00}(r) = C_{00}^2 r^{-1} + r^{-1} \int_0^r \xi f_{L00}(\xi) d\xi$ and $R'_{00}(L) = c_L$, one has

$$C_{00}^2 = Lc_L - \int_0^L \xi f_{L00}(\xi) d\xi. \quad (4.22)$$

On the other hand, it follows from the compatibility condition (4.6) that

$$c_L = \frac{1}{2\pi L(\frac{\pi}{2} - \theta_0)} \int_{Q_L} f_L(y) dy.$$

This, together with (4.13) and (4.22), yields $C_{00}^2 = 0$ holds. It is noted that $C_{00}^1 = R_{00}(0) = -\sum_{n=1}^{\infty} R_{0n}^{(2)}(0)$, then the solution of (4.17) can be expressed as

$$R_{00}(r) = -\sum_{n=1}^{\infty} R_{0n}^{(2)}(0) + \int_0^r \eta^{-1} \int_0^\eta \xi f_{L00}(\xi) d\xi d\eta, \quad (4.23)$$

here we especially point out that the numbers $R_{0n}^{(2)}(0)$ ($n = 1, 2, \dots$) in (4.23) have not been known yet.

Secondly, by the boundedness of $R_{m0}(0)$ for $m \in \mathbb{N}$, then

$$C_{m0}^2 = - \int_0^L \eta^{2\sqrt{\mu_m}-1} \int_\eta^L \xi^{1-\sqrt{\mu_m}} f_{Lm0}(\xi) d\xi d\eta.$$

In this case,

$$R_{m0}(r) = C_{m0}^1 r^{\sqrt{\mu_m}} - r^{-\sqrt{\mu_m}} \int_0^r \eta^{2\sqrt{\mu_m}-1} \int_\eta^L \xi^{1-\sqrt{\mu_m}} f_{Lm0}(\xi) d\xi d\eta$$

and

$$R'_{m0}(L) = C_{m0}^1 \sqrt{\mu_m} L^{\sqrt{\mu_m}-1} + \sqrt{\mu_m} L^{-\sqrt{\mu_m}-1} \int_0^L \eta^{2\sqrt{\mu_m}-1} \int_\eta^L \xi^{1-\sqrt{\mu_m}} f_{Lm0}(\xi) d\xi d\eta.$$

Together with the boundary condition $R'_{m0}(L) = 0$ in (4.18), this yields

$$C_{m0}^1 = -L^{-2\sqrt{\mu_m}} \int_0^L \eta^{2\sqrt{\mu_m}-1} \int_\eta^L \xi^{1-\sqrt{\mu_m}} f_{Lm0}(\xi) d\xi d\eta.$$

Consequently, the solution of (4.18) has the following expression

$$\begin{aligned} R_{m0}(r) &= -r^{\sqrt{\mu_m}} L^{-2\sqrt{\mu_m}} \int_0^L \eta^{2\sqrt{\mu_m}-1} \int_\eta^L \xi^{1-\sqrt{\mu_m}} f_{Lm0}(\xi) d\xi d\eta \\ &\quad - r^{-\sqrt{\mu_m}} \int_0^r \eta^{2\sqrt{\mu_m}-1} \int_\eta^L \xi^{1-\sqrt{\mu_m}} f_{Lm0}(\xi) d\xi d\eta. \end{aligned} \quad (4.24)$$

Thirdly, we solve (4.19). By the boundedness of $R_{0n}^{(i)}(0)$ for $i = 1, 2$ and the properties of Bessel functions as $r \rightarrow 0$ in Lemma 2.3, we can get from the expression of $R_{0n}^{(i)}(r)$

$$C_{0n}^{i2} = \lim_{r \rightarrow 0} \int_0^r s I_0(ns) f_{L0n}^{(i)}(s) ds = 0.$$

In addition, a simple computation shows

$$\left(R_{0n}^{(i)} \right)'(r) = C_{0n}^{i1} n I_0(nr) - n I_0'(nr) \int_r^L s K_0(ns) f_{L0n}^{(i)}(s) ds - n K_0'(nr) \int_0^r s I_0(ns) f_{L0n}^{(i)}(s) ds.$$

This, together with the boundary condition $(R_{0n}^{(i)})'(L) = 0$, yields

$$C_{0n}^{i1} = \frac{K_0'(nL)}{I_0'(nL)} \int_0^L s I_0(ns) f_{L0n}^{(i)}(s) ds.$$

Thus, the solution of (4.19) is

$$\begin{aligned} R_{0n}^{(i)}(r) &= \frac{K_0'(nL)}{I_0'(nL)} I_0(nr) \int_0^L s I_0(ns) f_{L0n}^{(i)}(s) ds - I_0(nr) \int_r^L s K_0(ns) f_{L0n}^{(i)}(s) ds \\ &\quad - K_0(nr) \int_0^r s I_0(ns) f_{L0n}^{(i)}(s) ds. \end{aligned} \quad (4.25)$$

Finally, we solve (4.20). It follows from $I_{\sqrt{\mu_m}}(0) = 0$, $K_{\sqrt{\mu_m}}(0) = \infty$, the boundedness of $R_{mn}^{(i)}(0)$ and the expression of $R_{mn}^{(i)}(r)$ in (4.21) that

$$C_{mn}^{i2} = 0.$$

Due to $(R_{mn}^{(i)})'(L) = 0$ and

$$(R_{mn}^{(i)})'(L) = C_{mn}^{i1} n I'_{\sqrt{\mu_m}}(nL) - n K'_{\sqrt{\mu_m}}(nL) \int_0^L s I_{\sqrt{\mu_m}}(ns) f_{Lmn}^{(i)}(s) ds,$$

one has

$$C_{mn}^{i1} = \frac{K'_{\sqrt{\mu_m}}(nL)}{I'_{\sqrt{\mu_m}}(nL)} \int_0^L s I_{\sqrt{\mu_m}}(ns) f_{Lmn}^{(i)}(s) ds.$$

Thus, we can obtain the solution to (4.20) as follows

$$\begin{aligned} R_{mn}^{(i)}(r) = & I_{\sqrt{\mu_m}}(nr) \left(\frac{K'_{\sqrt{\mu_m}}(nL)}{I'_{\sqrt{\mu_m}}(nL)} \int_0^L s I_{\sqrt{\mu_m}}(ns) f_{Lmn}^{(i)}(s) ds - \int_r^L s K_{\sqrt{\mu_m}}(ns) f_{Lmn}^{(i)}(s) ds \right) \\ & - K_{\sqrt{\mu_m}}(nr) \int_0^r s I_{\sqrt{\mu_m}}(ns) f_{Lmn}^{(i)}(s) ds. \end{aligned} \quad (4.26)$$

Collecting (4.23)-(4.26), the formal solution of (4.4) can be expressed as

$$v_L(r, \theta, y_2) = R_{00}(r) + I_1 + I_2 + I_3, \quad (4.27)$$

where

$$\begin{cases} I_1 = \sum_{m=1}^{\infty} R_{m0}(r) \cos \sqrt{\mu_m}(\theta - \theta_0), \\ I_2 = \sum_{n=1}^{\infty} \left(R_{0n}^{(1)}(r) \sin ny_2 + R_{0n}^{(2)}(r) \cos ny_2 \right), \\ I_3 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(R_{mn}^{(1)}(r) \cos \sqrt{\mu_m}(\theta - \theta_0) \sin ny_2 + R_{mn}^{(2)}(r) \cos \sqrt{\mu_m}(\theta - \theta_0) \cos ny_2 \right). \end{cases} \quad (4.28)$$

Step 2. The uniform convergence of (4.27)

In order to show that $v_L(r, \theta, y_2)$ in (4.27) is a real solution to (4.4), we require to give a more precise estimates on f_L . Since $f_L \in H_{4,\alpha}^{(1-\delta, 2-\delta_0)}$ and $f_L(r, \theta, y_2 + 2\pi) = f_L(r, \theta, y_2)$, which means that $\partial_{r,\theta,y_2}^4 f_L(r, \theta, y_2)$ is continuous for the variables $(r, \theta, y_2) \in (0, L] \times [\theta_0, \frac{\pi}{2}] \times [0, 2\pi]$, then we can use the integration by parts to obtain that for $n, m \geq 1$,

$$f_{Lm0}(r) = -\frac{(-1)^m}{\mu_m \pi (\frac{\pi}{2} - \theta_0)} \int_0^{2\pi} (\partial_\theta f_L(r, \frac{\pi}{2}, y_2) - \partial_\theta f_L(r, \theta_0, y_2)) dy_2 \\ - \frac{1}{\mu_m \pi (\frac{\pi}{2} - \theta_0)} \int_{\theta_0}^{\frac{\pi}{2}} \int_0^{2\pi} \partial_\theta^2 f_L(r, \theta, y_2) \cos \sqrt{\mu_m}(\theta - \theta_0) d\theta dy_2, \quad (4.29)$$

$$f_{L0n}^{(1)}(r) = \frac{1}{n^4 \pi (\frac{\pi}{2} - \theta_0)} \int_{\theta_0}^{\frac{\pi}{2}} \int_0^{2\pi} \partial_{y_2}^4 f_L(r, \theta, y_2) \sin ny_2 dy_2 d\theta, \quad (4.30)$$

$$f_{Lmn}^{(1)}(r) = -\frac{2(-1)^m}{n^2 \mu_m \pi (\frac{\pi}{2} - \theta_0)} \int_0^{2\pi} (\partial_\theta \partial_{y_2}^2 f_L(r, \frac{\pi}{2}, y_2) - \partial_\theta \partial_{y_2}^2 f_L(r, \theta_0, y_2)) \sin ny_2 dy_2 \\ + \frac{2}{n^2 \mu_m \pi (\frac{\pi}{2} - \theta_0)} \int_{\theta_0}^{\frac{\pi}{2}} \int_0^{2\pi} \partial_\theta^2 \partial_{y_2}^2 f_L(r, \theta, y_2) \sin ny_2 \cos \sqrt{\mu_m}(\theta - \theta_0) dy_2 d\theta. \quad (4.31)$$

From (4.29)-(4.31), we can derive that for $0 < r \leq 1$

$$|f_{L00}(r)| \leq C \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} r^{\delta-1} \quad (4.32)$$

and

$$\begin{cases} |f_{Lm0}(r)| \leq C \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} r^{\delta-1}, \\ |f_{Lm0}(r)| \leq C \mu_m^{-\frac{1}{2}} \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)} r^{\delta-1}, \\ |f_{Lm0}(r)| \leq C \mu_m^{-1} \|f\|_{2,\alpha;Q}^{(1-\delta,2-\delta_0)} r^{\delta-1} \end{cases} \quad (4.33)$$

and

$$|f_{L0n}^{(1)}(r)| \leq C n^{-k} \|f\|_{k,\alpha;Q}^{(1-\delta,2-\delta_0)} r^{\delta-1-k}, \quad k = 0, 1, 2, 3, \quad (4.34)$$

and

$$\begin{cases} |f_{Lmn}^{(1)}(r)| \leq C n^{-k} \|f\|_{k,\alpha;Q}^{(1-\delta,2-\delta_0)} r^{\delta-1-k}, \quad k = 0, 1, 2, \\ |f_{Lmn}^{(1)}(r)| \leq C n^{-1} \mu_m^{-1} \|f\|_{3,\alpha;Q}^{(1-\delta,2-\delta_0)} r^{\delta-2}, \\ |f_{Lmn}^{(1)}(r)| \leq C n^{-2} \mu_m^{-1} \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)} r^{\delta-3}, \end{cases} \quad (4.35)$$

for $r > 1$

$$|f_{L00}(r)| \leq C \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} r^{\delta_0-1} \quad (4.36)$$

and

$$\begin{cases} |f_{Lm0}(r)| \leq C \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} r^{\delta_0-2}, \\ |f_{Lm0}(r)| \leq C \mu_m^{-\frac{1}{2}} \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)} r^{\delta_0-2}, \\ |f_{Lm0}(r)| \leq C \mu_m^{-1} \|f\|_{2,\alpha;Q}^{(1-\delta,2-\delta_0)} r^{\delta_0-2} \end{cases} \quad (4.37)$$

and

$$|f_{L0n}^{(1)}(r)| \leq C n^{-k} \|f\|_{k,\alpha;Q}^{(1-\delta,2-\delta_0)} r^{\delta_0-2-k}, \quad k = 0, 1, 2, 3, \quad (4.38)$$

and

$$\begin{cases} |f_{Lmn}^{(1)}(r)| \leq C n^{-k} \|f\|_{k,\alpha;Q}^{(1-\delta,2-\delta_0)} r^{\delta_0-2-k}, \quad k = 0, 1, 2, \\ |f_{Lmn}^{(1)}(r)| \leq C n^{-1} \mu_m^{-1} \|f\|_{3,\alpha;Q}^{(1-\delta,2-\delta_0)} r^{\delta_0-3}, \\ |f_{Lmn}^{(1)}(r)| \leq C n^{-2} \mu_m^{-1} \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)} r^{\delta_0-4}. \end{cases} \quad (4.39)$$

We now start to show that the series in (4.28) are convergent for $(r, \theta, y_2) \in (0, L] \times [\theta_0, \frac{\pi}{2}] \times [0, 2\pi]$. In fact, by Lemma A.1-Lemma A.4 in Appendix (which are based on (4.32)-(4.39)), one has

$$\|I_1\|_{0,\alpha;Q_L}^{(0,-\delta_0)} \leq C \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)}. \quad (4.40)$$

$$\|I_2\|_{0,0;Q_L}^{(0,0)} \leq C \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)}, \quad (4.41)$$

$$\|R_{00}\|_{0,0;Q_L}^{(0,-\delta_0)} \leq C \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)}, \quad (4.42)$$

$$\|I_3\|_{0,0;Q_L}^{(0,0)} \leq C \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)}, \quad (4.43)$$

where $C > 0$ is independent of L .

Thus, combining (4.40)-(4.43) yields the uniform convergence of (4.27) for $(r, \theta, y_2) \in (0, L] \times [\theta_0, \frac{\pi}{2}] \times [0, 2\pi]$. Moreover,

$$\|v_L\|_{0,0;Q_L}^{(0,-\delta_0)} \leq C \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)}, \quad (4.44)$$

where $C > 0$ is independent of L .

Step 3. The convergence of $\nabla_{r,\theta,y_2} v_L$ and $\nabla_{r,\theta,y_2}^2 v_L$

We only give the proof on the convergence of $\partial_{y_2}^2 v_L$ since the other cases can be treated analogously. It follows from (4.27) and a direct computation that

$$\begin{aligned} \partial_{y_2}^2 v_L(r, \theta, y_2) = & - \sum_{n=1}^{\infty} n^2 \left(R_{0n}^{(1)}(r) \sin ny_2 + R_{0n}^{(2)}(r) \cos ny_2 \right) \\ & - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^2 \left(R_{mn}^{(1)}(r) \cos \sqrt{\mu_m}(\theta - \theta_0) \sin ny_2 + R_{mn}^{(2)}(r) \cos \sqrt{\mu_m}(\theta - \theta_0) \cos ny_2 \right). \end{aligned} \quad (4.45)$$

Thus, we have

$$|\partial_{y_2}^2 v_L(r, \theta, y_2)| \leq \sum_{n=1}^{\infty} n^2 \left(|R_{0n}^{(1)}(r)| + |R_{0n}^{(2)}(r)| \right) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^2 \left(|R_{mn}^{(1)}(r)| + |R_{mn}^{(2)}(r)| \right). \quad (4.46)$$

By Lemma A.5-Lemma A.6 in Appendix, we have that

$$\sum_{n=1}^{\infty} n^2 |R_{0n}^{(1)}(r)| \leq \begin{cases} C \|f\|_{3,\alpha;Q}^{(1-\delta,2-\delta_0)} r^{\delta-\frac{5}{2}}, & 0 < r \leq 1, \\ C \|f\|_{3,\alpha;Q}^{(1-\delta,2-\delta_0)}, & 1 < r \leq L \end{cases}$$

and

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^2 |R_{mn}^{(1)}(r)| \leq \begin{cases} C \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)} r^{\min\{\delta, \frac{1}{2}\}-\frac{5}{2}}, & r \leq 1, \\ C \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)}, & 1 < r \leq L, \end{cases}$$

where the generic $C > 0$ is independent of L .

Similarly, we also have

$$\sum_{n=1}^{\infty} n^2 |R_{0n}^{(2)}(r)| \leq \begin{cases} C \|f\|_{3,\alpha;Q}^{(1-\delta,2-\delta_0)} r^{\delta-\frac{5}{2}}, & 0 < r \leq 1, \\ C \|f\|_{3,\alpha;Q}^{(1-\delta,2-\delta_0)}, & 1 < r \leq L \end{cases}$$

and

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^2 |R_{mn}^{(2)}(r)| \leq \begin{cases} C \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)} r^{\min\{\delta, \frac{1}{2}\}-\frac{5}{2}}, & r \leq 1, \\ C \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)}, & 1 < r \leq L. \end{cases}$$

Thus, the series in (4.46) are convergent for any $(r, \theta, y_2) \in (0, L] \times [\theta_0, \pi] \times [0, 2\pi]$.

Collecting Step 1-Step 3, we complete the proof of Proposition 4.1. \square

§5. Higher regularities and existence of the solution to (4.2)

In this section, based on Proposition 4.1, we will establish the higher regularities of the solution v_L to (4.4) in the domain $Q_{\frac{L}{2}}$, subsequently, we show the solvability of (4.2) in the whole domain Q .

Lemma 5.1. *Suppose that v_L is a solution to (4.4), then the following estimate holds*

$$\|v_L\|_{6,\alpha;Q_{\frac{L}{2}}}^{(0,-\delta_0)} \leq C \|f_L\|_{4,\alpha;Q_{\frac{L}{2}}}^{(1-\delta,2-\delta_0)}. \quad (5.1)$$

Proof. We now apply the scaling technique to establish (5.1).

At first, set $y = Lz$, $\tilde{v}(z) = v_L(y)$ and $\tilde{f}(z) = f_L(y)$, then it follows from (4.4) and a direct computation that $\tilde{v}(z)$ satisfies

$$\Delta \tilde{v}(z) = L^2 \tilde{f}(z) \quad \text{in} \quad Q_1,$$

where $\tilde{v}(z)$ and $\tilde{f}(z)$ are $\frac{2\pi}{L}$ -periodic with respect to z_2 .

Denoting $Q(r_1, r_2) = \{y : y \in Q_{r_2}, r_1 < \sqrt{y_1^2 + y_3^2} < r_2\}$, and applying the standard Schauder interior estimate and boundary estimate (see Chapter 6 of [15]), one has

$$\|\tilde{v}(z)\|_{6,\alpha;Q(\frac{1}{3}, \frac{2}{3})} \leq C (\|\tilde{v}(z)\|_{0;Q(\frac{1}{4}, \frac{3}{4})} + \|L^2 \tilde{f}\|_{4,\alpha;Q(\frac{1}{4}, \frac{3}{4})}). \quad (5.2)$$

Going back to the function $v_L(y)$, we have that for any $y \in Q(\frac{L}{3}, \frac{2L}{3})$

$$\sum_{m=0}^6 L^m |D_y^m v_L| \leq C (\|v_L\|_{0;Q(\frac{L}{4}, \frac{3L}{4})} + \sum_{m=0}^4 L^{m+2} \|D_y^m f_L\|_{0;Q(\frac{L}{4}, \frac{3L}{4})} + L^{6+\alpha} [D_y^4 f_L]_{0,\alpha;Q(\frac{L}{4}, \frac{3L}{4})}). \quad (5.3)$$

Noticing that for any $y \in Q(\frac{L}{4}, \frac{3L}{4})$, we have $r_y \sim L$. Then multiplying $L^{-\delta_0}$ on the two hand sides of (5.3) yields for $L > 4$

$$\sum_{m=0}^6 r_y^{m-\delta} |D_y^m v_L(y)| \leq C (\|v_L\|_{0;Q(\frac{L}{4}, \frac{3L}{4})}^{(\star, -\delta_0)} + \|f_L\|_{4,\alpha;Q(\frac{L}{4}, \frac{3L}{4})}^{(\star, 2-\delta_0)}). \quad (5.4)$$

On the other hand, for $y, z \in Q(\frac{L}{3}, \frac{2L}{3})$, it follows from (5.2) that

$$L^{6+\alpha} \frac{|D^6 v_L(y) - D^6 v_L(z)|}{|y - z|^\alpha} \leq C(\|v_L\|_{0;Q(\frac{L}{4}, \frac{3L}{4})} + \sum_{m=0}^4 L^{m+2} \|D_y^m f_L\|_{0;Q(\frac{L}{4}, \frac{3L}{4})} + L^{6+\alpha} [D_y^4 f_L]_{0,\alpha;Q(\frac{L}{4}, \frac{3L}{4})}).$$

Similar to the proof of (5.4), we have

$$r_{y,z}^{6+\alpha-\delta_0} \frac{|D^6 v_L(y) - D^6 v_L(z)|}{|y - z|^\alpha} \leq C(\|v_L\|_{0;Q(\frac{L}{4}, \frac{3L}{4})}^{(\star, -\delta_0)} + \|f_L\|_{4,\alpha;Q(\frac{L}{4}, \frac{3L}{4})}^{(\star, 2-\delta_0)}). \quad (5.5)$$

Combining (5.4)-(5.5) with Proposition 4.1 yields

$$\|v_L\|_{6,\alpha;Q(\frac{L}{3}, \frac{2L}{3})}^{(\star, -\delta_0)} \leq C\|f_L\|_{4,\alpha;Q_L}^{(1-\delta, 2-\delta_0)}. \quad (5.6)$$

We now continue to prove (5.1). For any fixed point $y_0 = (y_1^0, y_2^0, y_3^0) \in Q_{\frac{L}{2}}$, we set $r_0 = \sqrt{(y_1^0)^2 + (y_3^0)^2}$, $d_0 = \mu r_0$ and the cylindrical domain $C_{d_0}(y_0) \equiv B((y_1^0, y_3^0), d_0) \times \mathbb{R}$, where $0 < \mu < 1$ is any fixed constant and $B((y_1^0, y_3^0), d_0)$ stands for a ball centered at (y_1^0, y_3^0) with the radius d_0 . We define the map $T : C_{d_0}(y_0) \rightarrow C_1(O) \equiv B((0, 0), 1) \times \mathbb{R}$ by $T(y) = \frac{y - y_0}{d_0}$ for $y \in C_{d_0}(y_0)$.

In order to estimate $v_L(y)$ in $Q_{\frac{L}{2}}$, we distinguish two cases:

- (i) $C_{d_0}(y_0) \subset Q_{\frac{L}{2}}$;
- (ii) $C_{d_0}(y_0) \cap \partial Q_{\frac{L}{2}} \neq \emptyset$.

We now treat these two cases separately. In case (i), we set $\bar{v}(x) = \frac{1}{d_0} v_L(y_0 + d_0 x)$ and $\bar{f}(x) = f(y_0 + d_0 x)$ for $x \in C_1(O)$. Then it follows that

$$\Delta \bar{v}(x) = d_0 \bar{f}(x),$$

where $\bar{v}(x)$ and $\bar{f}(x)$ are $\frac{2\pi}{d_0}$ -periodic with respect to the variable x_2 .

By the Schauder interior estimate in Chapter 6 of [15], one has

$$\|\bar{v}\|_{6,\alpha;C_{\frac{2}{3}}(O)} \leq C(\|\bar{v}\|_{0;C_1(O)} + \|d_0 \bar{f}\|_{4,\alpha;C_1(O)}), \quad (5.7)$$

where $C > 0$ depends only on α .

For $y \in C_{\frac{2d_0}{3}}(y_0)$, then $(\frac{1}{\mu} - \frac{2}{3})d_0 \leq r_y = \sqrt{y_1^2 + y_3^2} \leq (\frac{2}{3} + \frac{1}{\mu})d_0$ holds, and (5.7) means that

$$\begin{aligned} \sum_{m=0}^6 d_0^{m-1} |D_y^m v_L(y)| &\leq C(d_0^{-1} \|v_L\|_{0;C_{d_0}(y_0)} + \sum_{m=0}^4 d_0^{m+1} \|D_y^m f_L\|_{0;C_{d_0}(y_0)} \\ &\quad + d_0^{5+\alpha} [D_y^4 f_L]_{0,\alpha;C_{d_0}(y_0)}). \end{aligned} \quad (5.8)$$

If $r_0 \geq 1$, then multiplying $d_0^{1-\delta_0}$ on the two hand sides of (5.8) to obtain

$$\begin{aligned} \sum_{m=0}^6 d_0^{m-\delta_0} |D_y^m v_L(y)| &\leq C(d_0^{-\delta_0} \|v_L\|_{0;C_{d_0}(y_0)} + \sum_{m=0}^4 d_0^{m+2-\delta_0} \|D_y^m f_L\|_{0;C_{d_0}(y_0)} \\ &\quad + d_0^{6+\alpha-\delta_0} [D_y^4 f_L]_{0,\alpha;C_{d_0}(y_0)}). \end{aligned} \quad (5.9)$$

If $r_0 < 1$, then multiplying d_0 on the two hand sides of (5.8) yields

$$\sum_{m=0}^6 d_0^m |D_y^m v_L(y)| \leq C(\|v_L\|_{0;C_{d_0}(y_0)} + \sum_{m=0}^4 d_0^{m+2} \|D_y^m f_L\|_{0;C_{d_0}(y_0)} + d_0^{6+\alpha} [D_y^4 f_L]_{0,\alpha;C_{d_0}(y_0)}). \quad (5.10)$$

Noticing $r_y \sim d_0$ for any $y \in C_{\frac{2d_0}{3}}(y_0)$, then it follows from (5.9)-(5.10) and Proposition 4.1 that

$$\|v_L\|_{6;C_{\frac{2d_0}{3}}(y_0)}^{(0,-\delta_0)} \leq C_\mu (\|v_L\|_{0;Q_L}^{(0,-\delta_0)} + \|f_L\|_{4,\alpha;Q_L}^{(-1-\delta,2-\delta_0)}) \leq C_\mu \|f_L\|_{4,\alpha;Q_L}^{(-1-\delta,2-\delta_0)}. \quad (5.11)$$

On the other hand, if $x_0 \in C_{\frac{d_0}{2}}(y_0)$, then we can derive $r_{x_0} \sim d_0$ and $r_{x_0,y_0} = \min(r_{x_0}, r_{y_0}) \sim d_0$, and further obtain by (5.7) that

$$r_{x_0,y_0}^{6+\alpha-\delta_0} \frac{|D^6 v(x_0) - D^6 v(y_0)|}{|x_0 - y_0|^\alpha} \leq C_\mu (\|v_L\|_{0;Q_L}^{(0,-\delta_0)} + \|f_L\|_{4,\alpha;Q_L}^{(-1-\delta,2-\delta_0)}), \quad r_0 \geq 1, \quad (5.12)$$

$$r_{x_0,y_0}^{6+\alpha-\delta_0} \frac{|D^6 v(x_0) - D^6 v(y_0)|}{|x_0 - y_0|^\alpha} \leq C_\mu (\|v_L\|_{0;Q_L}^{(0,-\delta_0)} + \|f_L\|_{4,\alpha;Q_L}^{(-1-\delta,2-\delta_0)}), \quad r_0 < 1. \quad (5.13)$$

If $x_0 \notin C_{\frac{d_0}{2}}(y_0)$ but $x_0 \in C_{d_0}(y_0)$, then we have

$$r_{x_0,y_0}^{6+\alpha-\delta_0} \frac{|D^6 v(x_0) - D^6 v(y_0)|}{|x_0 - y_0|^\alpha} \leq C_\mu \sup_{y \in C_{d_0}(y_0)} r_y^{6-\delta_0} |D_y^6 v_L(y)|, \quad r_0 \geq 1, \quad (5.14)$$

$$r_{x_0,y_0}^{6+\alpha} \frac{|D^6 v(x_0) - D^6 v(y_0)|}{|x_0 - y_0|^\alpha} \leq C_\mu \sup_{y \in C_{d_0}(y_0)} r_y^6 |D_y^6 v_L(y)|, \quad r_0 < 1. \quad (5.15)$$

In case (ii), set $\bar{v}(y) = \frac{1}{d_0} v_L(y_0 + d_0 y)$ and $\bar{f}(y) = f_L(y_0 + d_0 y)$ for $y \in M \equiv T(C_{d_0}(y_0) \cap Q_L)$. As in case (i), since we already have shown $v_L \in C^{6,\alpha}(\overline{Q(\frac{L}{3}, \frac{2L}{3})})$, then it follows from the Schauder boundary estimate in Chapter 6 of [15] that

$$\|\bar{v}\|_{6,\alpha;C_{\frac{1}{2}}(O) \cap M} \leq C(\|\bar{v}\|_{0;M} + \|d_0 \bar{f}\|_{4,\alpha;M}), \quad (5.16)$$

where C depends only on α . Thus similar to the proof in case (i), one can obtain the similar estimates as in (5.11)-(5.15). Combining all estimates (i) and case (ii), we can derive that (5.1) holds. Thus, the proof of Lemma 5.1 is complete. \square

Next, we focus on improving the regularities of v_L near $r = 0$.

Lemma 5.2. *Suppose that v_L is the solution to (4.4), then the following estimate holds*

$$\|v_L\|_{6,\alpha;Q_{\frac{L}{2}}}^{(-1-\delta,-\delta_0)} \leq C \|f_L\|_{4,\alpha;Q_L}^{(1-\delta,2-\delta_0)}. \quad (5.17)$$

Proof. From Lemma 3.1 of [20], one has

$$|v_L|_{2+\alpha;Q_1}^{(-1-\delta)} \leq C (|f_L|_{\alpha;Q_1}^{(1-\delta)} + |v_L|_{0;Q_1}). \quad (5.18)$$

Then it follows from (4.8), (5.18) and Lemma 2.2 that

$$|v_L|_{2+\alpha;Q_1}^{(-1-\delta)} \leq C \|f_L\|_{4,\alpha;Q_L}^{(1-\delta,2-\delta_0)}. \quad (5.19)$$

On the other hand, it follows from (i)-(ii) in Lemma 2.2 and (5.19) that

$$|v_L|_{1+\delta;\overline{Q_1}} \leq |v_L|_{2+\alpha;Q_1}^{(-1-\delta)} \leq C \|f_L\|_{4,\alpha;Q_L}^{(1-\delta,2-\delta_0)}, \quad (5.20)$$

$$|v_L|_{2;Q_1}^{(-1-\delta)} \leq C |v_L|_{2+\alpha;Q_1}^{(-1-\delta)} \leq C \|f_L\|_{4,\alpha;Q_L}^{(1-\delta,2-\delta_0)}. \quad (5.21)$$

It follows from the boundary condition in (4.4) and the regularity of v_L in (5.20) that

$$\partial_{y_1} v_L(0, y_2, 0) = \partial_{y_3} v_L(0, y_2, 0) = 0. \quad (5.22)$$

From (5.20), we have for any $y \in Q_1$

$$\sup_{y \in Q_1} \frac{|\partial_{y_1} v_L(y) - \partial_{y_1} v_L(0, y_2, 0)|}{|y_1^2 + y_3^2|^{\frac{\delta}{2}}} \leq C \|f_L\|_{4,\alpha;Q_L}^{(1-\delta,2-\delta_0)},$$

which means

$$\sup_{y \in Q_1} |r_y^{-\delta} \partial_{y_1} v_L| \leq C \|f_L\|_{4,\alpha;Q_L}^{(1-\delta,2-\delta_0)}. \quad (5.23)$$

Similarly,

$$\sup_{y \in Q_1} |r_y^{-\delta} \partial_{y_3} v_L| \leq C \|f_L\|_{4,\alpha;Q_L}^{(1-\delta,2-\delta_0)}. \quad (5.24)$$

In addition, it follows from (5.21) that

$$\sup_{\sigma > 0} (\sigma^{1-\delta} \sup_{y \in Q_{1\sigma}} |D^2 v_L|) \leq C \|f_L\|_{4,\alpha;Q_L}^{(1-\delta,2-\delta_0)}. \quad (5.25)$$

Choosing $\sigma = \frac{r_y}{2}$, then by (5.25) we have for any $y \in Q_1$

$$\left(\frac{r_y}{2}\right)^{1-\delta} \sup_{z \in Q_{1\sigma}} |D^2 v_L(z)| \leq C \|f_L\|_{4,\alpha;Q_L}^{(1-\delta,2-\delta_0)}.$$

This, together with $r_y > \sigma$, yields

$$\sup_{y \in Q_1} r_y^{1-\delta} |D^2 v_L(y)| \leq C \|f_L\|_{4,\alpha;Q_L}^{(1-\delta,2-\delta_0)}. \quad (5.26)$$

Based on (5.26), we then derive the higher estimates near the y_2 -axis. Let $w_2(y) = \partial_{y_2}^2 v_L(y)$, then $w_2(y)$ satisfies following equation

$$\begin{cases} \Delta w_2 = \partial_{y_2}^2 f_L & \text{in } Q_L, \\ \partial_n w_2 = 0 & \text{on } \theta = \theta_0 \quad \text{and} \quad \theta = \pi, \\ w_2(y) = \partial_{y_2}^2 v_L(y) & \text{on } \sqrt{y_1^2 + y_2^2} = 1. \end{cases} \quad (5.27)$$

It follows from (5.27) and the scaling method used in Lemma 5.1 that

$$\|w_2\|_{4,\alpha;Q_1}^{(1-\delta,\star)} \leq C(\|w_2\|_{0;Q_1}^{(1-\delta,\star)} + \|\partial_{y_2}^2 f_L\|_{2,\alpha;Q_1}^{(3-\delta,\star)}).$$

This, together with (5.26), yields

$$\|\partial_{y_2}^2 v_L\|_{4,\alpha;Q_1}^{(1-\delta,\star)} \leq C\|f_L\|_{4,\alpha;Q_1}^{(1-\delta,\star)}. \quad (5.28)$$

Next, we focus on the estimates on $\partial_{y_1} v_L$. Let $w_1 = \partial_{y_1} v_L$, we now derive the boundary conditions of w_1 . On $y_1 = 0$, $w_1 = \partial_n v_L = 0$ holds. On $y_3 = b_0 y_1$, one has from (4.4) that

$$\partial_n v_L = \sin \theta_0 \partial_{y_1} v_L - \cos \theta_0 \partial_{y_3} v_L = 0. \quad (5.29)$$

Taking the tangent derivative $\cos \theta_0 \partial_{y_1} + \sin \theta_0 \partial_{y_3}$ on two hand sides of (5.29) yields

$$\begin{aligned} 0 &= (\cos \theta_0 \partial_{y_1} + \sin \theta_0 \partial_{y_3})(\sin \theta_0 \partial_{y_1} v_L - \cos \theta_0 \partial_{y_3} v_L) \\ &= \sin \theta_0 \cos \theta_0 \partial_{y_1}^2 v_L + (\sin^2 \theta_0 - \cos^2 \theta_0) \partial_{y_1 y_3}^2 v_L - \sin \theta_0 \cos \theta_0 \partial_{y_3}^2 v_L. \end{aligned} \quad (5.30)$$

In addition, it follows from (4.4) that

$$\partial_{y_3}^2 v_L = f_L - \partial_{y_1}^2 v_L - \partial_{y_2}^2 v_L. \quad (5.31)$$

Substituting (5.31) into (5.30) yields

$$2b_0 \partial_1 w_1 + (b_0^2 - 1) \partial_3 w_1 = b_0 f_L - b_0 \partial_{y_2}^2 v_L. \quad (5.32)$$

It is easy to verify

$$(2b_0, b_0^2 - 1) \cdot \vec{n} = (1 + b_0^2) \cos \theta_0 > 0.$$

Thus, $w_1(y)$ satisfies the following problem with the oblique derivative boundary condition on $\theta = \theta_0$

$$\begin{cases} \Delta w_1 = \partial_{y_1} f_L & \text{in } Q_L, \\ 2b_0 \partial_{y_1} w_1 + (b_0^2 - 1) \partial_{y_3} w_1 = b_0 f_L - b_0 \partial_{y_2}^2 v_L & \text{on } \theta = \theta_0, \\ w_1 = 0 & \text{on } \theta = \frac{\pi}{2}, \\ w_1(y) = \partial_{y_1} v_L(y) & \text{on } \sqrt{y_1^2 + y_3^2} = 1. \end{cases} \quad (5.33)$$

Still applying the scaling method as in Lemma 5.1, we have

$$\|w_1\|_{5,\alpha;Q_1}^{(-\delta,\star)} \leq C \left(\|w_1\|_{0;Q_1}^{(-\delta,\star)} + \|b_0 f_L - b_0 \partial_{y_2}^2 v_L\|_{4,\alpha;Q_1}^{(1-\delta,\star)} + \|\partial_{y_1} f_L\|_{3,\alpha;Q_1}^{(2-\delta,\star)} \right).$$

This, together with (5.23) and (5.28), yields

$$\|\partial_{y_1} v_L\|_{5,\alpha;Q_1}^{(-\delta,\star)} \leq C \|f_L\|_{4,\alpha;Q_L}^{(-1-\delta,2-\delta_0)}. \quad (5.34)$$

Similarly, for $\partial_{y_2} v_L$, we have same estimates as follows

$$\|\partial_{y_2} v_L\|_{5,\alpha;Q_1}^{(-\delta,\star)} \leq C \|f_L\|_{4,\alpha;Q_L}^{(-1-\delta,2-\delta_0)}. \quad (5.35)$$

Thus, combining (5.1), (5.20), (5.28), and (5.34)-(5.35), we obtain (5.17) and complete the proof of Lemma 5.2. \square

Finally, we start to prove the existence of the solution to (4.3), which also means the existence of the problem (4.2). At first, it follows from (4.5) and (5.17) that

$$\|v_L\|_{6,\alpha;Q_{\frac{L}{2}}}^{(-1-\delta,-\delta_0)} \leq C \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)}. \quad (5.36)$$

Suppose that \tilde{v}_L is an extension of v_L in the whole domain Q (one can see Theorem 7.25 of [15]), which satisfies

$$\tilde{v}_L|_{Q_{\frac{L}{2}}} = v_L, \quad \|\tilde{v}_L\|_{6,\alpha;Q}^{(-1-\delta,-\delta_0)} \leq C \|v_L\|_{6,\alpha;Q_{\frac{L}{2}}}^{(-1-\delta,-\delta_0)}. \quad (5.37)$$

Then it follows from (5.36)-(5.37) that

$$\|\tilde{v}_L\|_{6,\alpha;Q}^{(-1-\delta,-\delta_0)} \leq C \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)}. \quad (5.38)$$

Let $L \rightarrow +\infty$, by the standard diagonal method, we can extract a convergent subsequence \tilde{v}_{L_n} ($n \in \mathbb{N}$) and a function $v \in H_{6,\alpha;Q}^{(-1-\delta,-\delta_0)}$ such that

$$\|\tilde{v}_{L_n} - v\|_{6,\alpha;Q_N}^{(-1-\delta,-\delta_0)} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty,$$

where Q_N is any fixed sub-domain of Q . Moreover, it is easy to know that v is a solution of (4.3).

§6. The uniqueness of the solution to (4.3)

In this section, we focus on the uniqueness of the solution v to (4.3) since the existence of solution v to (4.3) has been shown in §5. To this end, we will use the separation method as in §4 together with some technical analysis so that the difficulty induced by the lack of the maximum principle for (4.3) can be overcome.

Lemma 6.1. *There exists a unique solution to (4.3) such that $v \in H_{6,\alpha}^{(-1-\delta,-\delta_0)}(Q)$.*

Proof. Suppose that $v_1(y), v_2(y) \in H_{6,\alpha}^{(-1-\delta,-\delta_0)}(Q)$ are different solutions to (4.3). Then for any $\delta_1 > \delta_0$, we have $v_1(y), v_2(y) \in H_{6,\alpha}^{(-1-\delta,-\delta_1)}(Q)$. Let $W(y) = v_1(y) - v_2(y)$,

then $W(y) \in H_{6,\alpha}^{(-1-\delta, -\delta_0)}(Q) \cap H_{6,\alpha}^{(-1-\delta, -\delta_1)}(Q)$. Denote $W_L(y)$ by the restriction of $W(y)$ on the domain Q_L , then we have

$$\begin{cases} \Delta W_L = 0 & \text{in } Q_L, \\ \partial_n W_L = 0 & \text{on } \theta = \theta_0 \text{ and } \theta = \pi, \\ W_L(y) = W_L(y_1, y_2 + 2\pi, y_3), \\ W_L(0, 0, 0) = 0, \\ \frac{\partial W_L}{\partial n}(y) = \frac{\partial v_1}{\partial n}(y) - \frac{\partial v_2}{\partial n}(y) = g(y) & \text{on } \sqrt{y_1^2 + y_3^2} = L. \end{cases} \quad (6.1)$$

Introducing the cylindrical (r, θ, y_2) as in (4.2), then (6.1) can be rewritten as

$$\begin{cases} \partial_r^2 W_L + r^{-2} \partial_\theta^2 W_L + \partial_{y_2}^2 W_L + r^{-1} \partial_r W_L = 0 & \text{in } Q_L, \\ \partial_n W_L = 0 & \text{on } \theta = \theta_0 \text{ and } \theta = \frac{\pi}{2}, \\ W_L(r, \theta, y_2) = W_L(r, \theta, y_2 + 2\pi), \\ W_L(0, \theta, 0) = 0, \\ \partial_r W_L(L, \theta, y_2) = g(L, \theta, y_2) & \text{on } \Sigma_L \equiv \{y : \sqrt{y_1^2 + y_3^2} = L, y_2 \in \mathbb{R}\}, \end{cases} \quad (6.2)$$

where $g(L, \theta, y_2)$ satisfies the compatibility conditions

$$\partial_\theta g(L, \theta, y_2) = \partial_\theta^3 g(L, \theta, y_2) = \partial_\theta^5 g(L, \theta, y_2) = 0 \quad \text{on } \theta = \theta_0 \text{ and } \theta = \frac{\pi}{2}, \quad (6.3)$$

and $\sup_{0 \leq \alpha_1 + \alpha_2 \leq 5} |L^{1-\delta_1+\alpha_1+\alpha_2} (\frac{1}{r} \partial_\theta)^{\alpha_1} \partial_{y_2}^{\alpha_2} g| \leq C$, which come from the regularity of $W(y) \in H_{6,\alpha}^{(-1-\delta, -\delta_1)}$ and

$$\sum_{0 \leq \alpha_1 + \alpha_2 \leq 5} \sup_{\substack{\theta \in [\theta_0, \frac{\pi}{2}] \\ y_2 \in \mathbb{R}}} L^{1-\delta_1+\alpha_2} |\partial_\theta^{\alpha_1} \partial_{y_2}^{\alpha_2} g| = \|g\|_{5, \Sigma_L}^{(1-\delta_1)} \leq CL^{\delta_0-\delta_1} \|g\|_{5, \Sigma_L}^{(1-\delta_0)}, \quad (6.4)$$

respectively. Moreover, the solvability condition holds

$$\int_0^{2\pi} \int_{\theta_0}^{\frac{\pi}{2}} g(L, \theta, y_2) d\theta dy_2 = 0. \quad (6.5)$$

By §4, it is known that the eigenvalues of the corresponding homogeneous problem of (6.2) are $\lambda_n = n^2 (n = 0, 1, \dots)$ and $\mu_m = (\frac{m\pi}{\frac{\pi}{2}-\theta_0})^2 (m = 0, 1, \dots)$, and the related complete orthogonal basis of eigenfunction functions is $\{\cos \sqrt{\mu_m}(\theta - \theta_0) \sin(ny_2), \cos \sqrt{\mu_m}(\theta - \theta_0) \cos(ny_2)\}_{m,n=0}^\infty$. Suppose that the expansion of $g(L, \theta, y_2)$ is

$$\begin{aligned} g(L, \theta, y_2) &= g_{00}(L) + \sum_{m=1}^\infty g_{m0}(L) \cos \sqrt{\mu_m}(\theta - \theta_0) + \sum_{n=1}^\infty \left(g_{0n}^{(1)}(L) \sin ny_2 + g_{0n}^{(2)}(L) \cos ny_2 \right) \\ &\quad + \sum_{m=1}^\infty \sum_{n=1}^\infty \left(g_{mn}^{(1)}(L) \cos \sqrt{\mu_m}(\theta - \theta_0) \sin ny_2 + g_{mn}^{(2)}(L) \cos \sqrt{\mu_m}(\theta - \theta_0) \cos ny_2 \right), \end{aligned} \quad (6.6)$$

where

$$\begin{aligned}
g_{00}(L) &= \frac{1}{2\pi(\frac{\pi}{2} - \theta_0)} \int_{\theta_0}^{\frac{\pi}{2}} \int_0^{2\pi} g(L, \theta, y_2) dy_2 d\theta, \\
g_{m0}(L) &= \frac{1}{\pi(\frac{\pi}{2} - \theta_0)} \int_{\theta_0}^{\frac{\pi}{2}} \int_0^{2\pi} g(L, \theta, y_2) \cos \sqrt{\mu_m}(\theta - \theta_0) dy_2 d\theta, \\
g_{0n}^{(1)}(L) &= \frac{1}{\pi(\frac{\pi}{2} - \theta_0)} \int_{\theta_0}^{\frac{\pi}{2}} \int_0^{2\pi} g(L, \theta, y_2) \sin ny_2 dy_2 d\theta, \\
g_{0n}^{(2)}(L) &= \frac{1}{\pi(\frac{\pi}{2} - \theta_0)} \int_{\theta_0}^{\frac{\pi}{2}} \int_0^{2\pi} g(L, \theta, y_2) \cos ny_2 dy_2 d\theta, \\
g_{mn}^{(1)}(L) &= \frac{2}{\pi(\frac{\pi}{2} - \theta_0)} \int_{\theta_0}^{\frac{\pi}{2}} \int_0^{2\pi} g(L, \theta, y_2) \cos \sqrt{\mu_m}(\theta - \theta_0) \sin ny_2 dy_2 d\theta, \\
g_{mn}^{(2)}(L) &= \frac{2}{\pi(\frac{\pi}{2} - \theta_0)} \int_{\theta_0}^{\frac{\pi}{2}} \int_0^{2\pi} g(L, \theta, y_2) \cos \sqrt{\mu_m}(\theta - \theta_0) \cos ny_2 dy_2 d\theta.
\end{aligned}$$

Let

$$\begin{aligned}
W_L(r, \theta, y_2) &= \tilde{R}_{00}(r) + \sum_{m=1}^{\infty} \tilde{R}_{m0}(r) \cos \sqrt{\mu_m}(\theta - \theta_0) + \sum_{n=1}^{\infty} \left(\tilde{R}_{0n}^{(1)}(r) \sin ny_2 + \tilde{R}_{0n}^{(2)}(r) \cos ny_2 \right) \\
&\quad + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\tilde{R}_{mn}^{(1)}(r) \cos \sqrt{\mu_m}(\theta - \theta_0) \sin ny_2 + \tilde{R}_{mn}^{(2)}(r) \cos \sqrt{\mu_m}(\theta - \theta_0) \cos ny_2 \right).
\end{aligned} \tag{6.7}$$

It follows from (6.2)-(6.3) and (6.7) that

$$\begin{cases} \tilde{R}_{00}''(r) + r^{-1} \tilde{R}_{00}'(r) = 0, \\ \tilde{R}_{00}'(L) = g_{00}(L) = 0, \\ \tilde{R}_{00}(0) = - \sum_{n=1}^{\infty} \tilde{R}_{0n}^{(2)}(0), \end{cases} \tag{6.8}$$

$$\begin{cases} \tilde{R}_{m0}''(r) - r^{-2} \mu_m \tilde{R}_{m0}(r) + r^{-1} \tilde{R}_{m0}'(r) = 0, & m = 1, 2, \dots, \\ \tilde{R}_{m0}'(L) = g_{m0}(L), \\ \tilde{R}_{m0}(0) \text{ is bounded, and } \tilde{R}_{m0}(0) + \sum_{n=1}^{\infty} R_{mn}^{(2)}(0) = 0, \end{cases} \tag{6.9}$$

$$\begin{cases} (\tilde{R}_{0n}^{(i)})''(r) - n^2 \tilde{R}_{0n}^{(i)}(r) + r^{-1} (\tilde{R}_{0n}^{(i)})'(r) = 0, & n = 1, 2, \dots, \\ (\tilde{R}_{0n}^{(i)})'(L) = g_{0n}^{(i)}(L), \\ \tilde{R}_{0n}^{(i)}(0) \text{ is bounded,} \end{cases} \tag{6.10}$$

and

$$\begin{cases} (\tilde{R}_{mn}^{(i)})''(r) - (r^{-2}\mu_m + n^2)\tilde{R}_{mn}^{(i)}(r) + r^{-1}(\tilde{R}_{mn}^{(i)})'(r) = 0, & m, n = 1, 2, \dots, \\ (\tilde{R}_{mn}^{(i)})'(L) = g_{mn}^{(i)}(L), \\ \tilde{R}_{mn}^{(i)}(0) \text{ is bounded,} \end{cases} \quad (6.11)$$

The general solution of (6.8) is

$$\tilde{R}_{00}(r) = C_{00}^1 + C_{00}^2 \ln r.$$

By the boundary condition $\tilde{R}'_{00}(L) = 0$, we have $C_{00}^2 = 0$ and $C_{00}^1 = -\sum_{n=1}^{\infty} \tilde{R}_{0n}^{(2)}(0)$. Thus

$$\tilde{R}_{00}(r) = -\sum_{n=1}^{\infty} \tilde{R}_{0n}^{(2)}(0). \quad (6.12)$$

For the equation (6.9), its general solution is

$$\tilde{R}_{m0}(r) = C_{m0}^1 r^{\sqrt{\mu_m}} + C_{m0}^2 r^{-\sqrt{\mu_m}}.$$

It is noted that the boundedness of $\tilde{R}_{m0}(0)$ implies $C_{m0}^2 = 0$ and $\tilde{R}'_{m0}(L) = g_{m0}(L)$ derives $C_{m0}^1 = \frac{g_{m0}(L)}{\sqrt{\mu_m}} L^{1-\sqrt{\mu_m}}$, then

$$\tilde{R}_{m0}(r) = \frac{g_{m0}(L)}{\sqrt{\mu_m}} L^{1-\sqrt{\mu_m}} r^{\sqrt{\mu_m}}. \quad (6.13)$$

In addition, the general solution of (6.11) is

$$\tilde{R}_{mn}^{(i)}(r) = C_{mn}^{i1} I_{\sqrt{\mu_m}}(nr) + C_{mn}^{i2} K_{\sqrt{\mu_m}}(nr), \quad i = 1, 2.$$

Due to the boundedness of $\tilde{R}_{mn}^{(i)}(0)$ and $(\tilde{R}_{mn}^{(i)})'(L) = g_{mn}^{(i)}(L)$, we get

$$C_{mn}^{i2} = 0, \quad C_{mn}^{i1} = \frac{g_{mn}^{(i)}(L)}{n I'_{\sqrt{\mu_m}}(nL)},$$

and then

$$\tilde{R}_{mn}^{(i)}(r) = \frac{g_{mn}^{(i)}(L)}{n I'_{\sqrt{\mu_m}}(nL)} I_{\sqrt{\mu_m}}(nr) \quad \text{for } m = 0, 1, \dots, n = 1, 2, \dots. \quad (6.14)$$

Here we have used $\tilde{R}_{0n}^{(i)}(0) = \frac{g_{0n}^{(i)}(L)}{n I'_0(nL)}$ and $\tilde{R}_{mn}^{(i)}(0) = 0$.

Combining (6.12)-(6.14), we get a formal solution of (6.2) as follows

$$\begin{aligned}
W_L(r, \theta, y_2) = & \tilde{R}_{00}(r) + \sum_{m=1}^{\infty} \tilde{R}_{m0}(r) \cos \sqrt{\mu_m}(\theta - \theta_0) + \sum_{n=1}^{\infty} \left(\tilde{R}_{0n}^{(1)}(r) \sin ny_2 + \tilde{R}_{0n}^{(2)}(r) \cos ny_2 \right) \\
& + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\tilde{R}_{mn}^{(1)}(r) \cos \sqrt{\mu_m}(\theta - \theta_0) \sin ny_2 + \tilde{R}_{mn}^{(2)}(r) \cos \sqrt{\mu_m}(\theta - \theta_0) \cos ny_2 \right),
\end{aligned} \tag{6.15}$$

where

$$\left\{ \begin{aligned} \tilde{R}_{00}(r) &= - \sum_{n=1}^{\infty} \tilde{R}_{0n}^{(2)}(0), \\ \tilde{R}_{m0}(r) &= \frac{g_{m0}(L)}{\sqrt{\mu_m}} L^{1-\sqrt{\mu_m}} r^{\sqrt{\mu_m}}, \quad m = 1, 2, \dots, \\ \tilde{R}_{mn}^{(i)}(r) &= \frac{g_{mn}^{(i)}(L)}{n I'_{\sqrt{\mu_m}}(nL)} I_{\sqrt{\mu_m}}(nr), \quad i = 1, 2, \quad m = 0, 1, \dots, n = 1, 2, \dots \end{aligned} \right. \tag{6.16}$$

Next, we show that the series in (6.15) is uniformly convergent in Q_L . To prove the convergence, we require to establish some estimates on the coefficients in (6.6) as in §4. It follows from a direct computation that

$$\begin{aligned}
g_{m0}(L) &= \frac{1}{\pi(\frac{\pi}{2} - \theta_0)} \int_{\theta_0}^{\frac{\pi}{2}} \int_0^{2\pi} g(L, \theta, y_2) \cos \sqrt{\mu_m}(\theta - \theta_0) dy_2 d\theta \\
&= - \frac{1}{\sqrt{\mu_m} \pi(\frac{\pi}{2} - \theta_0)} \int_{\theta_0}^{\frac{\pi}{2}} \int_0^{2\pi} \partial_{\theta} g(L, \theta, y_2) \sin \sqrt{\mu_m}(\theta - \theta_0) dy_2 d\theta,
\end{aligned} \tag{6.17}$$

$$\begin{aligned}
g_{0n}^{(1)}(L) &= \frac{1}{\pi(\frac{\pi}{2} - \theta_0)} \int_{\theta_0}^{\frac{\pi}{2}} \int_0^{2\pi} g(L, \theta, y_2) \sin ny_2 dy_2 d\theta \\
&= \frac{1}{n\pi(\frac{\pi}{2} - \theta_0)} \int_{\theta_0}^{\frac{\pi}{2}} \int_0^{2\pi} \partial_{y_2} g(L, \theta, y_2) \cos ny_2 dy_2 d\theta,
\end{aligned} \tag{6.18}$$

and

$$\begin{aligned}
g_{mn}^{(1)}(L) &= \frac{2}{\pi(\frac{\pi}{2} - \theta_0)} \int_{\theta_0}^{\frac{\pi}{2}} \int_0^{2\pi} g(L, \theta, y_2) \cos \sqrt{\mu_m}(\theta - \theta_0) \sin ny_2 dy_2 d\theta \\
&= - \frac{2}{\sqrt{\mu_m} \pi(\frac{\pi}{2} - \theta_0)} \int_{\theta_0}^{\frac{\pi}{2}} \int_0^{2\pi} \partial_{\theta} g(L, \theta, y_2) \sin \sqrt{\mu_m}(\theta - \theta_0) \sin ny_2 dy_2 d\theta \\
&= \frac{2}{\mu_m \pi(\frac{\pi}{2} - \theta_0)} \int_{\theta_0}^{\frac{\pi}{2}} \int_0^{2\pi} \partial_{\theta}^2 g(L, \theta, y_2) \cos \sqrt{\mu_m}(\theta - \theta_0) \sin ny_2 dy_2 d\theta \\
&= \frac{2}{n\mu_m \pi(\frac{\pi}{2} - \theta_0)} \int_{\theta_0}^{\frac{\pi}{2}} \int_0^{2\pi} \partial_{y_2} \partial_{\theta}^2 g(L, \theta, y_2) \cos \sqrt{\mu_m}(\theta - \theta_0) \cos ny_2 dy_2 d\theta.
\end{aligned} \tag{6.19}$$

From (6.17)-(6.19), we arrive at

$$\begin{cases} |g_{m0}(L)| \leq C\mu_m^{-\frac{1}{2}} \|g\|_{1,0;\Sigma_L}^{(1-\delta_1)} L^{\delta_1-1}, \\ |g_{0n}^{(1)}(L)| \leq Cn^{-1} \|g\|_{1,0;\Sigma_L}^{(1-\delta_1)} L^{\delta_1-2}, \\ |g_{mn}^{(1)}(L)| \leq Cn^{-1} \mu_m^{-1} \|g\|_{3,0;\Sigma_L}^{(1-\delta_1)} L^{\delta_1-2}. \end{cases} \quad (6.20)$$

Analogously, (6.20) are also true for $|g_{0n}^{(2)}(L)|$ and $|g_{mn}^{(2)}(L)|$.

We now show that the series in the expression of $W_L(r, \theta, y_2)$ are convergent for any fixed point $(r, \theta, y_2) \in (0, L] \times [\theta_0, \frac{\pi}{2}] \times [0, 2\pi]$. At first, by (6.16) and (6.20), we get

$$\begin{aligned} \sum_{m=1}^{\infty} |\tilde{R}_{m0}(r)| &\leq \sum_{m=1}^{\infty} \frac{|g_{m0}(L)|}{\sqrt{\mu_m}} L^{1-\sqrt{\mu_m}} r^{\sqrt{\mu_m}} \\ &\leq C \|g\|_{1,0;\Sigma_L}^{(1-\delta_1)} \sum_{m=1}^{\infty} \mu_m^{-1} L^{\delta_1-\sqrt{\mu_m}} r^{\sqrt{\mu_m}} \\ &\leq C \|g\|_{1,0;\Sigma_L}^{(1-\delta_1)} r^{\delta_1} \sum_{m=1}^{\infty} \mu_m^{-1} \\ &\leq C \|g\|_{1,0;\Sigma_L}^{(1-\delta_1)} r^{\delta_1}, \end{aligned} \quad (6.21)$$

which derives

$$\begin{cases} \sum_{m=1}^{\infty} |\tilde{R}_{m0}(r)| \leq C \|g\|_{1,0;\Sigma_L}^{(1-\delta_1)}, & r \leq 1; \\ r^{-\delta_1} \sum_{m=1}^{\infty} |\tilde{R}_{m0}(r)| \leq C \|g\|_{1,0;\Sigma_L}^{(1-\delta_1)}, & r \geq 1. \end{cases} \quad (6.22)$$

Next, we use the properties of modified Bessel functions in Lemma 2.3 and (6.20) to show the convergence of the series $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\tilde{R}_{mn}^{(1)}(r)|$. To this end, the following three cases will be considered separately, where M will represent a suitably large fixed integer.

Case (a) $m < M$ and $nr < 1$

$$\begin{aligned} \sum_{m < M} \sum_{nr < 1} |\tilde{R}_{mn}^{(1)}(r)| &\leq \sum_{m < M} \sum_{nr < 1} \frac{|g_{mn}^{(1)}(L)|}{n I'_{\sqrt{\mu_m}}(nL)} I_{\sqrt{\mu_m}}(nr) \\ &\leq \sum_{m < M} \sum_{nr < 1} n^{-1} \frac{\sqrt{2\pi nL}}{e^{nL}} \frac{e^{nr} (nr)^{\sqrt{\mu_m}}}{2\sqrt{\mu_m} \Gamma(\sqrt{\mu_m} + 1)} |g_{mn}^{(1)}(L)| \\ &\leq C \|g\|_{2,0;\Sigma_L}^{(1-\delta_1)} \sum_{m < M} \sum_{nr < 1} n^{-\frac{3}{2}} e^{-n} 2^{-\sqrt{\mu_m}} \mu_m^{-\frac{1}{2}} L^{\delta_1-\frac{3}{2}} \\ &\leq C \|g\|_{2,0;\Sigma_L}^{(1-\delta_1)}. \end{aligned} \quad (6.23)$$

Case (b) $m < M$ and $1 \leq nr \leq nL$

$$\begin{aligned}
\sum_{m < M} \sum_{1 \leq nr \leq nL} |\tilde{R}_{mn}^{(1)}(r)| &\leq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{|g_{mn}^{(1)}(L)|}{nI'_{\sqrt{\mu_m}}(nL)} I_{\sqrt{\mu_m}}(nr) \\
&\leq C \|g\|_{3,0;\Sigma_L}^{(1-\delta_1)} \sum_{m < M} \sum_{1 \leq nr \leq nL} n^{-1} \frac{\sqrt{2\pi nL}}{e^{nL}} \frac{e^{nr}}{\sqrt{2\pi nr}} n^{-1} \mu_m^{-1} L^{\delta_1-2} \\
&\leq C \|g\|_{3,0;\Sigma_L}^{(1-\delta_1)} L^{\delta_1-\frac{3}{2}} \sum_{m < M} \sum_{1 \leq nr \leq nL} n^{-\frac{3}{2}} \mu_m^{-1} \\
&\leq C \|g\|_{3,0;\Sigma_L}^{(1-\delta_1)}. \tag{6.24}
\end{aligned}$$

Case (c) $m > M$

It follows from a direct computation and Lemma 2.3 that

$$\begin{aligned}
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\tilde{R}_{mn}^{(1)}(r)| &\leq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{|g_{mn}^{(1)}(L)|}{nI'_{\sqrt{\mu_m}}(nL)} I_{\sqrt{\mu_m}}(nr) \\
&\leq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^{-1} |g_{mn}^{(1)}(L)| \frac{z(L)}{(1+z^2(L))^{\frac{1}{4}}} \frac{1}{(1+z^2(r))^{\frac{1}{4}}} e^{\sqrt{\mu_m}(\eta(r)-\eta(L))} \\
&\leq C \|g\|_{3,0;\Sigma_L}^{(1-\delta_1)} L^{\delta_1-\frac{3}{2}} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^{-\frac{3}{2}} \mu_m^{-\frac{3}{4}} \\
&\leq C \|g\|_{3,0;\Sigma_L}^{(1-\delta_1)}. \tag{6.25}
\end{aligned}$$

Thus, collecting (6.23)-(6.25) yields

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\tilde{R}_{mn}^{(1)}(r)| \leq C \|g\|_{3,0;\Sigma_L}^{(1-\delta_1)}. \tag{6.26}$$

Finally, we prove the convergence of the series $\sum_{n=1}^{\infty} |\tilde{R}_{0n}^{(1)}(r)|$. In fact, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} |\tilde{R}_{0n}^{(1)}(r)| &\leq \sum_{n=1}^{\infty} \frac{|g_{0n}^{(1)}(L)|}{nI'_0(nL)} I_0(nr) \\
&\leq C \|g\|_{1,0;\Sigma_L}^{(1-\delta_1)} L^{\delta_1-2} \sum_{n=1}^{\infty} n^{-2} \frac{\sqrt{2\pi nL}}{e^{nL}} e^{nr} \\
&\leq C \|g\|_{1,0;\Sigma_L}^{(1-\delta_1)} L^{\delta_1-\frac{3}{2}} \sum_{n=1}^{\infty} n^{-\frac{3}{2}} \\
&\leq C \|g\|_{1,0;\Sigma_L}^{(1-\delta_1)}, \tag{6.27}
\end{aligned}$$

which implies

$$|\tilde{R}_{00}(r)| \leq \sum_{n=1}^{\infty} |\tilde{R}_{0n}^{(2)}(0)| \leq C \|g\|_{1,0;\Sigma_L}^{(1-\delta_1)}. \tag{6.28}$$

By (6.21) and (6.26)-(6.28), we have established the convergence of $W_L(r, \theta, y_2)$ in Q_L and the following estimates

$$\begin{cases} |W_L(r, \theta, y_2)| \leq C \|g\|_{3,0;\Sigma_L}^{(1-\delta_1)}, & r \leq 1; \\ r^{-\delta_1} |W_L(r, \theta, y_2)| \leq C \|g\|_{3,0;\Sigma_L}^{(1-\delta_1)}, & r \geq 1. \end{cases}$$

And hence

$$\|W_L\|_{0,0;Q_L}^{(0,-\delta_1)} \leq C \|g\|_{3,0;\Sigma_L}^{(1-\delta_1)}, \quad (6.29)$$

where the constant $C > 0$ is independent of L .

Let $L \rightarrow +\infty$, then one has from (6.4) that

$$\|g\|_{3,0;\Sigma_L}^{(1-\delta_1)} \rightarrow 0. \quad (6.30)$$

Combining (6.29) with (6.30) yields

$$\|W\|_{0,0;Q}^{(0,-\delta_1)} = \lim_{L \rightarrow +\infty} \|W_L\|_{0,0;Q_L}^{(0,-\delta_1)} = 0.$$

Thus, the proof of Lemma 6.1 is complete. \square

Going back to (4.2), based on Lemma 5.1 and Lemma 6.1 we have

Proposition 6.2. *Suppose $\dot{f} \in H_{4,\alpha}^{(1-\delta,2-\delta_0)}(Q)$, $\dot{g}_i \in H_{5,\alpha}^{(-\delta,1-\delta_0)}$ ($i = 1, 2$), then there exists a unique solution \dot{u} to (4.2), which satisfies the following estimate*

$$\|\dot{u}\|_{6,\alpha;Q}^{(-1-\delta,-\delta_0)} \leq C \left(\|\dot{f}\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)} + \sum_{i=1}^2 \|\dot{g}_i\|_{5,\alpha;Q}^{(-\delta,1-\delta_0)} \right). \quad (6.31)$$

§7. Proofs of Theorem 3.1 and Theorem 1.1.

In this section, first we will use the contraction mapping principle to show Theorem 3.1. To this end, we define the space $K = \{v \in C(\bar{Q}) : v - u_0 \in H_{6,\alpha}^{(-1-\delta,-\delta_0)}, \|v - u_0\|_{6,\alpha;Q}^{(-1-\delta,-\delta_0)} \leq \varepsilon\}$. Set $u = \dot{u} + u_0$, where \dot{u} is defined as the solution to the following linearized problem which is analogous to the problem (4.1) in §4

$$\begin{cases} \Delta \dot{u} = \dot{F} & \text{in } Q, \\ \partial_n \dot{u} = \dot{G}_1 & \text{on } y_3 = b_0 y_1, \\ \partial_{y_1} \dot{u} = \dot{G}_2 & \text{on } y_1 = 0, \\ \dot{u}(0, 0, 0) = 0, \\ \lim_{y_1+y_3 \rightarrow \infty} |\nabla_y \dot{u}| = 0, \end{cases} \quad (7.1)$$

where

$$\begin{aligned} \dot{F}(v, \nabla_y v, \nabla_y^2 v) &= \Delta \dot{v} - L(v, \nabla_y v, \nabla_y^2 v) \dot{v}, \\ \dot{G}_1(v, \nabla_y v) &= (G_1(u_0, \nabla_y u_0)u_0 - G_1(v, \nabla_y v)u_0) + \frac{\partial \dot{v}}{\partial n} - G_1(v, \nabla_y v) \dot{v}, \\ \dot{G}_2(v, \nabla_y v) &= (G_2(u_0, \nabla_y u_0)u_0 - G_2(v, \nabla_y v)u_0) + \partial_{y_1} \dot{v} - G_2(v, \nabla_y v) \dot{v}, \end{aligned}$$

with $\dot{v} = v - u_0$. Denote the mapping J by $J(v) = u$, then we have the following lemma.

Lemma 7.1. *Suppose that the positive constants α , δ and δ_0 ($0 < \alpha, \delta, \delta_0 < 1$) are given in Proposition 6.2, then there exists an ε_0 such that for $\varepsilon \in (0, \varepsilon_0)$, J is a mapping from K to itself.*

Proof. Set

$$\dot{F}(v, \nabla_y v, \nabla_y^2 v) \equiv \sum_{i=1}^7 I_i, \quad (7.2)$$

where

$$\begin{aligned} I_1 &= -\frac{1}{q_0} \sum_{i,j=1}^3 a_{ij}(\nabla_x \varphi^-, \nabla_y v) \partial_{x_i x_j}^2 \varphi^-, \\ I_{i+1} &= (1 - A_{ii}(v, \nabla_y v)) \partial_{y_i}^2 \dot{v}, \quad i = 1, 2, 3, \\ I_5 &= -2A_{12}(v, \nabla_y v) \partial_{y_1 y_2}^2 \dot{v}, \\ I_6 &= -2A_{13}(v, \nabla_y v) \partial_{y_1 y_3}^2 \dot{v}, \\ I_7 &= -2A_{23}(v, \nabla_y v) \partial_{y_2 y_3}^2 \dot{v}. \end{aligned}$$

We now treat each I_i separately. It follows from $\nabla^2 \Phi_0 = 0$ and Lemma 2.4 that

$$\|a_{ij}(\nabla_x \varphi^-, \nabla_y v) \partial_{x_i x_j}^2 \varphi^-\|_{4,\alpha;Q}^{(1-\delta, 2-\delta_0)} \leq \|a_{ij}(\nabla_x \varphi^-, \nabla_y v)\|_{4,\alpha;Q}^{(0,0)} \|\partial_{x_i x_j}^2 \varphi^-\|_{6,\alpha;Q}^{(1-\delta, 2-\delta_0)} \leq C\varepsilon,$$

then we have

$$\|I_1\|_{6,\alpha;Q}^{(-1-\delta, -\delta_0)} \leq Cq_0^{-1}\varepsilon. \quad (7.3)$$

To analyze I_2 , we rewrite $I_2 = \sum_{i=1}^6 I_2^i$ with

$$\begin{aligned} I_2^1 &= \left(1 - \frac{a_{11}}{\partial_{y_1} v + b_0 \partial_{y_3} v}\right) \partial_1^2 \dot{v}, \\ I_2^2 &= -\frac{a_{22}(\partial_{y_2} v)^2}{(\partial_{y_1} v + b_0 \partial_{y_3} v)^3} \partial_1^2 \dot{v}, \\ I_2^3 &= -\frac{a_{33}(\partial_{y_3} v)^2}{(\partial_{y_1} v + b_0 \partial_{y_3} v)^3} \partial_1^2 \dot{v}, \\ I_2^4 &= \frac{2a_{12} \partial_{y_2} v}{(\partial_{y_1} v + b_0 \partial_{y_3} v)^2} \partial_1^2 \dot{v}, \\ I_2^5 &= \frac{2a_{13} \partial_{y_3} v}{(\partial_{y_1} v + b_0 \partial_{y_3} v)^2} \partial_1^2 \dot{v}, \\ I_2^6 &= -\frac{2a_{23} \partial_{y_2} v \partial_{y_3} v}{(\partial_{y_1} v + b_0 \partial_{y_3} v)^2} \partial_1^2 \dot{v}. \end{aligned}$$

Notice that

$$\begin{aligned}
& \|1 - a_{11}(\nabla_x \varphi^-, \nabla_y v)\|_{4,\alpha;Q}^{(0,0)} = \left\| \frac{(\partial_{x_1} \varphi^- - \partial_{x_1} \Phi)^2}{c^2(\nabla_x \varphi^-, \nabla_y v)} \right\|_{4,\alpha;Q}^{(0,0)} \\
& \leq \left\| \frac{(\partial_{x_1} \varphi^- - \partial_{x_1} \Phi)^2}{c^2(\nabla_x \varphi^-, \nabla_y v)} - \frac{(\partial_{x_1} \varphi_0^- - \partial_{x_1} \Phi_0)^2}{c^2(\nabla_x \varphi^-, \nabla_y v)} \right\|_{4,\alpha;Q}^{(0,0)} + \left\| \frac{(\partial_{x_1} \varphi_0^- - \partial_{x_1} \Phi_0)^2}{c^2(\nabla_x \varphi^-, \nabla_y u_0)} \right\|_{4,\alpha;Q}^{(0,0)} \\
& \leq \left\| \frac{(\partial_{x_1} \varphi^- - \partial_{x_1} \Phi)^2}{c^2(\nabla_x \varphi^-, \nabla_y v) c^2(\nabla_x \varphi_0^-, \nabla_y u_0)} \right\|_{4,\alpha;Q}^{(0,0)} \|c^2(\nabla_x \varphi^-, \nabla_y v) - c^2(\nabla_x \varphi_0^-, \nabla_y u_0)\|_{4,\alpha;Q}^{(-\delta, 1-\delta_0)} \\
& \quad + \left\| \frac{(\partial_{x_1} \varphi^- - \partial_{x_1} \Phi) + (\partial_{x_1} \varphi_0^- - \partial_{x_1} \Phi_0)}{c^2(\nabla_x \varphi_0^-, \nabla_y u_0)} \right\|_{4,\alpha;Q}^{(0,0)} \left(\|\partial_{x_1} \varphi^- - \partial_{x_1} \varphi_0^-\|_{4,\alpha;Q}^{(-\delta, 1-\delta_0)} \right. \\
& \quad \left. + \|\partial_{x_1} \Phi - \partial_{x_1} \Phi_0\|_{4,\alpha;Q}^{(-\delta, 1-\delta_0)} \right) + C q_0^{-\frac{4}{\gamma-1}} \\
& \leq C(q_0^{-1} + q_0^{-\frac{2}{\gamma-1}})(q_0 \varepsilon + \varepsilon) + C q_0^{-\frac{4}{\gamma-1}}, \tag{7.4}
\end{aligned}$$

and similarly,

$$\|1 - a_{ii}(\nabla_x \varphi^-, \nabla_y v)\|_{4,\alpha;Q}^{(0,0)} \leq C(q_0^{-1} + q_0^{-\frac{2}{\gamma-1}})(q_0 \varepsilon + \varepsilon) + C q_0^{-\frac{4}{\gamma-1}}, \quad i = 2, 3, \tag{7.5}$$

$$\|a_{ij}(\nabla_x \varphi^-, \nabla_y v)\|_{4,\alpha;Q}^{(0,0)} \leq C(q_0^{-1} + q_0^{-\frac{2}{\gamma-1}})(q_0 \varepsilon + \varepsilon) + C q_0^{-\frac{4}{\gamma-1}}, \quad 1 \leq i \neq j \leq 3. \tag{7.6}$$

In addition, one has

$$\partial_{y_1} u_0 = 1 + O(q_0^{-\frac{2}{\gamma-1}}), \quad \partial_{y_2} u_0 = 0, \quad \partial_{y_3} u_0 = O(q_0^{-\frac{2}{\gamma-1}}), \quad \nabla_{y_i y_j}^2 u_0 = 0, \quad 1 \leq i, j \leq 3,$$

this yields together with (7.4)

$$\|I_2^1\|_{4,\alpha;Q}^{(1-\delta, 2-\delta_0)} \leq C(q_0^{-\frac{2}{\gamma-1}} + \varepsilon)\varepsilon,$$

and analogously,

$$\|I_2^i\|_{4,\alpha;Q}^{(1-\delta, 2-\delta_0)} \leq C(q_0^{-\frac{2}{\gamma-1}} + \varepsilon)\varepsilon, \quad i = 2, \dots, 6.$$

Therefore, we have

$$\|I_2\|_{4,\alpha;Q}^{(1-\delta, 2-\delta_0)} \leq C(q_0^{-\frac{2}{\gamma-1}} + \varepsilon)\varepsilon. \tag{7.7}$$

By the same method, we can arrive at

$$\|I_i\|_{4,\alpha;Q}^{(1-\delta, 2-\delta_0)} \leq C(q_0^{-\frac{2}{\gamma-1}} + \varepsilon)\varepsilon, \quad i = 3, \dots, 7. \tag{7.8}$$

Thus, substituting (7.3) and (7.7)-(7.8) into (7.2) yields

$$\|\dot{F}(v, \nabla_y v, \nabla_y^2 v)\|_{4,\alpha;Q}^{(1-\delta, 2-\delta_0)} \leq C(q_0^{-\frac{2}{\gamma-1}} + \varepsilon)\varepsilon. \tag{7.9}$$

On the other hand, it follows from a direct computation that

$$\begin{aligned}\dot{G}_1(v, \nabla_y v) &= \frac{1}{q_0} (\sin \theta_0 \partial_{y_3} u_0 + \cos \theta_0 \partial_{y_1} u_0) ((\partial_{x_3} \varphi - \partial_{x_3} \varphi_0) - b_0 (\partial_{x_1} \varphi^- - \partial_{x_1} \varphi_0^-)) \\ &\quad + \frac{\sin \theta_0}{q_0} (b_0 (q_0 - \partial_{x_1} \varphi^-) + \partial_{x_3} \varphi^-) \partial_{y_3} \dot{v} + \frac{\cos \theta_0}{q_0} \partial_{x_3} \varphi^- \partial_{y_1} \dot{v} - \frac{\sin \theta_0}{q_0} (\partial_{x_1} \varphi^- - q_0) \partial_{y_1} \dot{v},\end{aligned}$$

which yields

$$\|\dot{G}_1(v, \nabla_y v)\|_{5, \alpha}^{(-\delta, 1-\delta_0)} \leq C q_0^{-1} \varepsilon. \quad (7.10)$$

Analogously,

$$\|\dot{G}_2(v, \nabla_y v)\|_{5, \alpha}^{(-\delta, 1-\delta_0)} \leq C q_0^{-1} \varepsilon. \quad (7.11)$$

For appropriately large q_0 and small $\varepsilon > 0$, then Proposition 6.2 implies that there exists a unique solution $\dot{u} \in H_{6, \alpha}^{(-1-\delta, -\delta_0)}$ to (7.1) such that

$$\|\dot{u}\|_{6, \alpha; Q}^{(-1-\delta, -\delta_0)} \leq C(q_0^{-1} + \varepsilon) \varepsilon \leq \varepsilon,$$

which means that mapping J is from K to itself. \square

Lemma 7.2. *Under the assumptions of Lemma 7.1, the mapping J is a contractible mapping from K to itself.*

Proof. Taking $v_1, v_2 \in K$. Let $u_i = Jv_i$ and $\dot{u}_i = u_i - u_0$ in Q , then we have

$$\begin{cases} \Delta(u_2 - u_1) = \dot{F}(v_1, \nabla_y v_1, \nabla_y^2 v_1) - \dot{F}(v_2, \nabla_y v_2, \nabla_y^2 v_2) & \text{in } Q, \\ \partial_n(u_2 - u_1) = \dot{G}_1(v_1, \nabla_y v_1) - \dot{G}_1(v_2, \nabla_y v_2) & \text{on } y_3 = b_0 y_1, \\ \partial_{y_1}(u_2 - u_1) = \dot{G}_2(v_1, \nabla_y v_1) - \dot{G}_2(v_2, \nabla_y v_2) & \text{on } y_1 = 0, \\ (u_2 - u_1)(0, 0, 0) = 0, \\ \lim_{y_1 + y_3 \rightarrow \infty} |\nabla_y(u_2 - u_1)| = 0. \end{cases} \quad (7.12)$$

As in Lemma 7.1, a direct computation yields

$$\begin{aligned}\|\dot{F}(v_2, \nabla_y v_2, \nabla_y^2 v_2) - \dot{F}(v_1, \nabla_y v_1, \nabla_y^2 v_1)\|_{4, \alpha; Q}^{(1-\delta, 2-\delta_0)} &\leq C \varepsilon \|v_2 - v_1\|_{6, \alpha; Q}^{(-1-\delta, -\delta_0)} \\ &\quad + C(q_0^{-1} + q_0^{-\frac{2}{\gamma-1}} + \varepsilon) \|u_2 - u_1\|_{6, \alpha; Q}^{(-1-\delta, -\delta_0)}\end{aligned} \quad (7.13)$$

and

$$\begin{aligned}&\|\dot{G}_i(v_2, \nabla_y v_2) - \dot{G}_i(v_1, \nabla_y v_1)\|_{5, \alpha; Q}^{(-\delta, 1-\delta_0)} \\ &\leq C \varepsilon \|v_2 - v_1\|_{6, \alpha; Q}^{(-1-\delta, -\delta_0)} + C(q_0^{-1} + q_0^{-\frac{2}{\gamma-1}} + \varepsilon) \|u_2 - u_1\|_{6, \alpha; Q}^{(-1-\delta, -\delta_0)}, \quad i = 1, 2.\end{aligned} \quad (7.14)$$

By Proposition 6.2, we have

$$\|u_2 - u_1\|_{6, \alpha; Q}^{(-1-\delta, -\delta_0)} \leq C \varepsilon \|v_2 - v_1\|_{6, \alpha; Q}^{(-1-\delta, -\delta_0)} + C(q_0^{-1} + q_0^{-\frac{2}{\gamma-1}} + \varepsilon) \|u_2 - u_1\|_{6, \alpha; Q}^{(-1-\delta, -\delta_0)}. \quad (7.15)$$

Choosing appropriately large q_0 and small ε_0 yields

$$\|Jv_2 - Jv_1\|_{6, \alpha; Q}^{(-1-\delta, -\delta_0)} \leq \frac{1}{2} \|v_2 - v_1\|_{6, \alpha; Q}^{(-1-\delta, -\delta_0)},$$

which means that J is a contractible mapping. \square

We now prove Theorem 3.1.

Proof of Theorem 3.1. By Lemma 7.1 and Lemma 7.2, we know that the mapping $u = Jv$ has a unique fixed point in the space $H_{6,\alpha}^{(-1-\delta, -\delta_0)}(Q)$, which implies that Theorem 3.1 is shown. \square

Based on Theorem 3.1, we can show Theorem 1.1.

Proof of Theorem 1.1. By Theorem 3.1, one knows that the problem (1.19) admits a unique solution $u \in H_{6,\alpha}^{(-1-\delta, -\delta_0)}(Q)$. Since only the condition $u(0, y_2^0, 0) = 0$ other than $u(0, y_2, 0) \equiv 0$ for all $y_2 \in \mathbb{R}$ is applied in order to solve the 3-D attached strong oblique shock problem (1.18), then (1.18) is obviously overdetermined. \square

Appendix

Lemma A.1. For the term I_1 defined in (4.28), we have

$$\|I_1\|_{0,\alpha;Q_L}^{(0,-\delta_0)} \leq C \|f\|_{0,\alpha;Q}^{(1-\delta, 2-\delta_0)}, \quad (A.1)$$

where the generic positive constant C is independent of L .

Proof. To estimate $\|I_1\|_{0,\alpha}^{(0,-\delta_0)}$, by the definition of $\|I_1\|_{0,\alpha}^{(0,-\delta_0)}$, we need to study the two cases including $0 < r \leq 1$ and $r \geq 1$ separately.

Case i. $0 < r \leq 1$

By the first inequalities in (4.33) and (4.37), and the fact of $0 < \delta < \sqrt{\mu_1} - 1$, then we get

$$\begin{aligned} & \sum_{m=1}^{\infty} |R_{m0}(r)| \\ & \leq \sum_{m=1}^{\infty} r^{\sqrt{\mu_m}} L^{-2\sqrt{\mu_m}} \left\{ \int_0^1 \eta^{2\sqrt{\mu_m}-1} \left(\int_{\eta}^1 \xi^{1-\sqrt{\mu_m}} |f_{Lm0}(\xi)| d\xi \right. \right. \\ & \quad \left. \left. + \int_1^L \xi^{1-\sqrt{\mu_m}} |f_{Lm0}(\xi)| d\xi \right) d\eta + \int_1^L \eta^{2\sqrt{\mu_m}-1} \int_{\eta}^L \xi^{1-\sqrt{\mu_m}} |f_{Lm0}(\xi)| d\xi d\eta \right\} \\ & \quad + r^{-\sqrt{\mu_m}} \int_0^r \eta^{2\sqrt{\mu_m}-1} \left(\int_{\eta}^1 \xi^{1-\sqrt{\mu_m}} |f_{Lm0}(\xi)| d\xi + \int_1^L \xi^{1-\sqrt{\mu_m}} |f_{Lm0}(\xi)| d\xi \right) d\eta \\ & \leq C \|f\|_{0,\alpha;Q}^{(1-\delta, 2-\delta_0)} \sum_{m=1}^{\infty} \left\{ r^{\sqrt{\mu_m}} L^{-2\sqrt{\mu_m}} \left[\int_0^1 \eta^{2\sqrt{\mu_m}-1} \left(\int_{\eta}^1 \xi^{\delta-\sqrt{\mu_m}} d\xi + \int_1^L \xi^{\delta_0-1-\sqrt{\mu_m}} d\xi \right) d\eta \right. \right. \\ & \quad \left. \left. + \int_1^L \eta^{2\sqrt{\mu_m}-1} \int_{\eta}^L \xi^{\delta_0-1-\sqrt{\mu_m}} d\xi d\eta \right] + r^{-\sqrt{\mu_m}} \int_0^r \eta^{2\sqrt{\mu_m}-1} \left(\int_{\eta}^1 \xi^{\delta-\sqrt{\mu_m}} d\xi \right. \right. \\ & \quad \left. \left. + \int_1^L \xi^{\delta_0-1-\sqrt{\mu_m}} d\xi \right) d\eta \right\} \end{aligned}$$

$$\begin{aligned}
&\leq C\|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \sum_{m=1}^{\infty} \left\{ r^{\sqrt{\mu_m}} L^{-2\sqrt{\mu_m}} \left[\int_0^1 \eta^{2\sqrt{\mu_m}-1} \left(\frac{\eta^{1+\delta-\sqrt{\mu_m}} - 1}{\sqrt{\mu_m} - \delta - 1} + \frac{1 - L^{\delta_0-\sqrt{\mu_m}}}{\sqrt{\mu_m} - \delta_0} \right) d\eta \right. \right. \\
&\quad \left. \left. + \int_1^L \eta^{2\sqrt{\mu_m}-1} \frac{\eta^{\delta_0-\sqrt{\mu_m}} - L^{\delta_0-\sqrt{\mu_m}}}{\sqrt{\mu_m} - \delta_0} d\eta \right] + r^{-\sqrt{\mu_m}} \int_0^r \eta^{2\sqrt{\mu_m}-1} \left(\frac{\eta^{1+\delta-\sqrt{\mu_m}} - 1}{\sqrt{\mu_m} - \delta - 1} \right. \right. \\
&\quad \left. \left. + \frac{1 - L^{\delta_0-\sqrt{\mu_m}}}{\sqrt{\mu_m} - \delta_0} \right) d\eta \right\} \\
&\leq C\|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \sum_{m=1}^{\infty} \left\{ \frac{r^{\sqrt{\mu_m}} L^{-2\sqrt{\mu_m}} + r^{1+\delta}}{(\sqrt{\mu_m} - \delta - 1)(\sqrt{\mu_m} + \delta + 1)} + \frac{r^{\sqrt{\mu_m}} (L^{-2\sqrt{\mu_m}} + 1)}{2\sqrt{\mu_m}(\sqrt{\mu_m} - \delta_0)} \right. \\
&\quad \left. + \frac{r^{\sqrt{\mu_m}} L^{\delta_0-\sqrt{\mu_m}}}{(\sqrt{\mu_m} - \delta_0)(\sqrt{\mu_m} + \delta_0)} \right\} \\
&\leq C\|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \sum_{m=1}^{\infty} \left\{ \frac{1}{\mu_m - (\delta + 1)^2} + \frac{1}{\sqrt{\mu_m}(\sqrt{\mu_m} - \delta_0)} + \frac{1}{\mu_m - \delta_0^2} \right\} \\
&\leq C\|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)}. \tag{A.2}
\end{aligned}$$

Case ii. $r \geq 1$

In this case, we have

$$\begin{aligned}
|R_{m0}(r)| &\leq r^{\sqrt{\mu_m}} L^{-2\sqrt{\mu_m}} \int_0^L \eta^{2\sqrt{\mu_m}-1} \int_{\eta}^L \xi^{1-\sqrt{\mu_m}} |f_{m0}(\xi)| d\xi d\eta \\
&\quad + r^{-\sqrt{\mu_m}} \int_0^r \eta^{2\sqrt{\mu_m}-1} \int_{\eta}^L \xi^{1-\sqrt{\mu_m}} |f_{m0}(\xi)| d\xi d\eta \\
&\equiv A_{m1} + A_{m2}.
\end{aligned}$$

For A_{m1} with $m \in \mathbb{N}$, by the same method as in Case i, we can get

$$\begin{aligned}
\sum_{m=1}^{\infty} A_{m1} &\leq C\|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \sum_{m=1}^{\infty} \left(\frac{r^{\sqrt{\mu_m}} L^{-2\sqrt{\mu_m}}}{\mu_m - (\delta + 1)^2} + \frac{r^{\sqrt{\mu_m}} L^{-2\sqrt{\mu_m}}}{2\sqrt{\mu_m}(\sqrt{\mu_m} - \delta_0)} + \frac{r^{\sqrt{\mu_m}} L^{\delta_0-\sqrt{\mu_m}}}{\mu_m - \delta_0^2} \right) \\
&\leq C\|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \sum_{m=1}^{\infty} \left(\frac{1}{\mu_m - (\delta + 1)^2} + \frac{1}{\sqrt{\mu_m}(\sqrt{\mu_m} - \delta_0)} + \frac{r^{\delta_0}}{\mu_m - \delta_0^2} \right) \\
&\leq C\|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} r^{\delta_0}. \tag{A.3}
\end{aligned}$$

For A_{m2} with $m \in \mathbb{N}$,

$$\begin{aligned}
&\sum_{m=1}^{\infty} A_{m2} \\
&\leq C\|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \sum_{m=1}^{\infty} r^{-\sqrt{\mu_m}} \left\{ \int_0^1 \eta^{2\sqrt{\mu_m}-1} \left(\int_{\eta}^1 \xi^{1-\sqrt{\mu_m}} \xi^{\delta-1} d\xi + \int_1^L \xi^{1-\sqrt{\mu_m}} \xi^{\delta_0-2} d\xi \right) d\eta \right. \\
&\quad \left. + \int_1^r \eta^{2\sqrt{\mu_m}-1} \int_{\eta}^L \xi^{1-\sqrt{\mu_m}} \xi^{\delta_0-2} d\xi d\eta \right\}
\end{aligned}$$

$$\begin{aligned}
&\leq C \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \sum_{m=1}^{\infty} r^{-\sqrt{\mu_m}} \left\{ \int_0^1 \eta^{2\sqrt{\mu_m}-1} \left(\frac{\eta^{1+\delta-\sqrt{\mu_m}} - 1}{\sqrt{\mu_m} - \delta - 1} + \frac{1 - L^{\delta_0-\sqrt{\mu_m}}}{\sqrt{\mu_m} - \delta_0} \right) d\eta \right. \\
&\quad \left. + \int_1^r \eta^{2\sqrt{\mu_m}-1} \frac{\eta^{\delta_0-\sqrt{\mu_m}} - L^{\delta_0-\sqrt{\mu_m}}}{\sqrt{\mu_m} - \delta_0} d\eta \right\} \\
&\leq C \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \sum_{m=1}^{\infty} \left\{ \frac{r^{-\sqrt{\mu_m}}}{(\sqrt{\mu_m} - \delta - 1)(\sqrt{\mu_m} + \delta + 1)} + \frac{r^{-\sqrt{\mu_m}}}{2\sqrt{\mu_m}(\sqrt{\mu_m} - \delta_0)} \right. \\
&\quad \left. + \frac{r^{\delta_0}}{(\sqrt{\mu_m} - \delta_0)(\sqrt{\mu_m} + \delta_0)} \right\} \\
&\leq C \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} r^{\delta_0} \sum_{m=1}^{\infty} \left(\frac{1}{\mu_m - (\delta + 1)^2} + \frac{1}{\sqrt{\mu_m}(\sqrt{\mu_m} - \delta_0)} + \frac{1}{\mu_m - \delta_0^2} \right) \\
&\leq C \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} r^{\delta_0}. \tag{A.4}
\end{aligned}$$

Thus, collecting the estimates (A.2)-(A.4) yields Lemma A.1. \square

Lemma A.2. *For the term I_2 defined in (4.28), we have*

$$\|I_2\|_{0,0;Q_L}^{(0,0)} \leq C \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)},$$

where $C > 0$ is independent of L .

Proof. By the expression of I_2 , it suffices to only treat $\sum_{n=1}^{\infty} |R_{0n}^{(1)}(r)|$ since $\sum_{n=1}^{\infty} |R_{0n}^{(2)}(r)|$ can be analogously estimated. We write

$$R_{0n}^{(1)}(r) = B_1^n - B_2^n, \tag{A.5}$$

where

$$\begin{aligned}
B_1^n &= \frac{K'_0(nL)}{I'_0(nL)} I_0(nr) \int_0^L s I_0(ns) f_{L0n}^{(i)}(s) ds, \\
B_2^n &= I_0(nr) \int_r^L s K_0(ns) f_{L0n}^{(i)}(s) ds + K_0(nr) \int_0^r s I_0(ns) f_{L0n}^{(i)}(s) ds.
\end{aligned}$$

Next we deal with $\sum_{n=1}^{\infty} |B_1^n|$ and $\sum_{n=1}^{\infty} |B_2^n|$ respectively.

By (iii) and (v) in Lemma 2.3, and the inequalities in (4.34) and (4.38), we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} |B_1^n| &\leq \sum_{n=1}^{\infty} \frac{\sqrt{\frac{\pi}{2nL}} e^{-nL}}{\frac{e^{nL}}{\sqrt{2\pi nL}}} e^{nr} \left(\int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^1 + \int_1^L \right) s I_0(ns) |f_{L0n}^{(1)}(s)| ds \\
&\leq C \sum_{n=1}^{\infty} e^{nr-2nL} \left\{ \int_0^{\frac{1}{n}} e^{ns} s^{\delta} ds \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} + \left(\int_{\frac{1}{n}}^1 e^{ns} s^{\delta-1} ds \right. \right. \\
&\quad \left. \left. + \int_1^L e^{ns} s^{\delta_0-2} ds \right) n^{-1} \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)} \right\}
\end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{n=1}^{\infty} e^{nr-2nL} \left\{ n^{-1-\delta} (e-1) \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} + \left(n^{-1-\delta} (e^n - e) \right. \right. \\
&\quad \left. \left. + n^{-2} (e^{nL} - e^n) \right) \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)} \right\} \\
&\leq \sum_{n=1}^{\infty} \left\{ n^{-1-\delta} e^{-nL} \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} + (n^{-1-\delta} e^{n(1-L)} + n^{-2}) \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)} \right\} \\
&\leq C \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)}. \tag{A.6}
\end{aligned}$$

To estimate $\sum_{n=1}^{\infty} |B_2^n|$, we will divide this procedure into the following three cases.

Case i. $0 < nr \leq 1$.

In this case, it follows from (iii) in Lemma 2.3 that

$$\begin{aligned}
\sum_{nr \leq 1} |B_2^n| &\leq \sum_{nr \leq 1} \left\{ e^{nr} \left(\int_r^{\frac{1}{n}} + \int_{\frac{1}{n}}^1 + \int_1^L \right) s K_0(ns) |f_{L_{0n}}^{(1)}(s)| ds \right. \\
&\quad \left. + K_0(nr) \int_0^r s I_0(ns) |f_{L_{0n}}^{(1)}(s)| ds \right\} \\
&\leq C \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \sum_{nr \leq 1} \left\{ e^{nr} \left(\int_r^{\frac{1}{n}} \frac{e^{-ns}}{\sqrt{2ns}} s^\delta ds + \int_{\frac{1}{n}}^1 \frac{e^{-ns}}{\sqrt{2ns}} s^\delta ds + \int_1^L \frac{e^{-ns}}{\sqrt{2ns}} s^{\delta_0-1} ds \right) \right. \\
&\quad \left. + \frac{e^{-nr}}{\sqrt{2nr}} \int_0^r e^{ns} s^\delta ds \right\} \\
&\leq C \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \sum_{nr \leq 1} \left(\frac{n^{-1-\delta}}{\delta + \frac{1}{2}} + \frac{n^{-1-\min(\frac{1}{2},\delta)}}{e} + \frac{n^{-1-\frac{1}{2}}}{e^n} + \frac{n^{-1-\delta}}{1+\delta} \right) \\
&\leq C \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)}. \tag{A.7}
\end{aligned}$$

Case ii. $1 \leq nr$ and $r \leq 1$

We get

$$\begin{aligned}
\sum_{1 \leq nr \leq n} |B_2^n| &\leq \sum_{1 \leq nr \leq n} \left\{ e^{nr} \left(\int_r^1 K_0(ns) |f_{L_{0n}}^{(1)}(s)| ds + \int_1^L s K_0(ns) |f_{L_{0n}}^{(1)}(s)| ds \right) \right. \\
&\quad \left. + K_0(nr) \left(\int_0^{\frac{1}{n}} s I_0(ns) |f_{L_{0n}}^{(1)}(s)| ds + \int_{\frac{1}{n}}^r s I_0(ns) |f_{L_{0n}}^{(1)}(s)| ds \right) \right\} \\
&\leq C \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \sum_{1 \leq nr \leq n} \left\{ e^{nr} \left(n^{-\frac{1}{2}} \int_r^1 e^{-ns} s^{\delta-\frac{1}{2}} ds + n^{-\frac{1}{2}} \int_1^L e^{-ns} s^{\delta_0-\frac{3}{2}} ds \right) \right. \\
&\quad \left. + \frac{e^{-nr}}{\sqrt{2nr}} \left(\int_0^{\frac{1}{n}} e^{ns} s^\delta ds + \int_{\frac{1}{n}}^r e^{ns} s^\delta ds \right) \right\}
\end{aligned}$$

$$\begin{aligned}
&\leq C \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \sum_{1 \leq nr \leq n} \left\{ \left(n^{-1-\min(\frac{1}{2},\delta)} + n^{-\frac{3}{2}} \right) + \left(e^{-nr} n^{-1-\delta} e + r^{\delta-\frac{1}{2}} n^{-\frac{3}{2}} \right) \right\} \\
&\leq C \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)}. \tag{A.8}
\end{aligned}$$

Case iii. $1 \leq r \leq L$.

At this time, we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} |B_2^n| &\leq \sum_{n=1}^{\infty} \left\{ e^{nr} \int_r^L K_0(ns) |f_{L_{0n}}^{(1)}(s)| ds + K_0(nr) \left(\int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^1 + \int_1^r \right) s I_0(ns) |f_{L_{0n}}^{(1)}(s)| ds \right\} \\
&\leq C \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \sum_{n=1}^{\infty} \left\{ e^{nr} \int_r^L \frac{e^{-ns}}{\sqrt{2ns}} s^{\delta_0-1} ds + \frac{e^{-nr}}{\sqrt{2nr}} \left(\int_0^{\frac{1}{n}} e^{ns} s^{\delta} + \int_{\frac{1}{n}}^1 e^{ns} s^{\delta} ds \right. \right. \\
&\quad \left. \left. + \int_1^r e^{ns} s^{\delta_0-1} ds \right) \right\} \\
&\leq C \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \sum_{n=1}^{\infty} n^{-\frac{1}{2}} \left\{ e^{nr} \int_r^L e^{-ns} s^{\delta_0-\frac{3}{2}} ds + e^{-nr} \left(\int_0^{\frac{1}{n}} e^{ns} s^{\delta} ds + \int_{\frac{1}{n}}^1 e^{ns} s^{\delta} ds \right. \right. \\
&\quad \left. \left. + \int_1^r e^{ns} s^{\delta_0-1} ds \right) \right\} \\
&\leq C \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \sum_{n=1}^{\infty} (n^{-\frac{3}{2}} + n^{-\delta-\frac{3}{2}} e^{-n}) \\
&\leq C \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)}. \tag{A.9}
\end{aligned}$$

Consequently, collecting (A.6)-(A.9) yields Lemma A.2. \square

Lemma A.3. For the term $R_{00}(r)$ defined in (4.27), we have

$$\|R_{00}\|_{0,0;Q_L}^{(0,-\delta_0)} \leq C \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)}, \tag{A.10}$$

where $C > 0$ is independent of L .

Proof. Noting that from (4.23)

$$|R_{00}(r)| \leq \sum_{n=1}^{\infty} |R_{0n}^{(2)}(0)| + \int_0^r \eta^{-1} \int_0^{\eta} \xi |f_{L_{00}}(\xi)| d\xi d\eta \equiv A_{01} + A_{02}.$$

By $I_0(0) = 1$ and Lemma A.2, we know that

$$\begin{aligned}
A_{01} &\leq \sum_{n=1}^{\infty} \left(\left| \frac{K'_0(nL)}{I'_0(nL)} \right| \int_0^L s I_0(ns) |f_{L_{0n}}^{(2)}(s)| ds + \int_r^L s K_0(ns) |f_{L_{0n}}^{(2)}(s)| ds \right) \\
&\leq C \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)}. \tag{A.11}
\end{aligned}$$

To estimate A_{02} , we divide this process into the following two cases.

Case i: $0 < r \leq 1$

By (4.32), we have

$$\begin{aligned}
A_{02} &\leq C \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \int_0^r \eta^{-1} \int_0^\eta \xi^\delta d\xi d\eta \\
&\leq C \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \frac{r^{1+\delta}}{(1+\delta)^2} \\
&\leq C \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)}.
\end{aligned} \tag{A.12}$$

Case ii: $1 < r \leq L$

By (4.36), we get

$$\begin{aligned}
A_{02} &\leq \int_0^1 \eta^{-1} \int_0^\eta \xi |f_{L00}(\xi)| d\xi d\eta + \int_1^r \eta^{-1} \left(\int_0^1 \xi |f_{L00}(\xi)| d\xi + \int_1^\eta \xi |f_{L00}(\xi)| d\xi \right) d\eta \\
&\leq C \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \left\{ \int_0^1 \eta^{-1} \int_0^\eta \xi^\delta d\xi d\eta + \int_1^r \eta^{-1} \left(\int_0^1 \xi^\delta d\xi + \int_1^\eta \xi^{\delta_0-1} d\xi \right) d\eta \right\} \\
&\leq C \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \left(\frac{1}{(1+\delta)^2} + \frac{\ln r}{(1+\delta)^2} + \frac{r^{\delta_0}}{\delta_0^2} \right) \\
&\leq C \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} r^{\delta_0}.
\end{aligned} \tag{A.13}$$

Thus, combining (A.11)-(A.13) yields (A.10). \square

Lemma A.4. *For the term I_3 defined in (4.28), we have*

$$\|I_3\|_{0,0;Q_L}^{(0,0)} \leq C \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)}, \tag{A.14}$$

where $C > 0$ is independent of L .

Proof. Since it follows from (vii)-(viii) in Lemma 2.3 that the Bessel functions $I_{\sqrt{\mu_m}}(x)$ and $K_{\sqrt{\mu_m}}(x)$ have different properties for $m \leq M$ and $m > M$ ($M \in \mathbb{N}$ is some suitably large integer), then we require to divide the process of estimating I_3 into the following four cases.

Case i. $m \leq M$, and $nr \leq 1$ with $0 < r \leq 1$.

At this time, by (iv)-(vii) in Lemma 2.3 and the inequalities in (4.35) and (4.39), we derive that

$$\begin{aligned}
|R_{mn}^{(1)}(r)| &\leq I_{\sqrt{\mu_m}}(nr) \left(\frac{K'_{\sqrt{\mu_m}}(nL)}{I'_{\sqrt{\mu_m}}(nL)} \int_0^L s I_{\sqrt{\mu_m}}(ns) |f_{Lmn}^{(1)}(s)| ds + \int_r^L s K_{\sqrt{\mu_m}}(ns) |f_{Lmn}^{(1)}(s)| ds \right) \\
&\quad + K_{\sqrt{\mu_m}}(nr) \int_0^r s I_{\sqrt{\mu_m}}(ns) |f_{Lmn}^{(1)}(s)| ds
\end{aligned}$$

$$\begin{aligned}
&\leq C \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \frac{e^{nr(\frac{nr}{2})\sqrt{\mu_m}}}{\Gamma(\sqrt{\mu_m}+1)} \left\{ e^{-2nL} \left[\int_0^{\frac{1}{n}} \frac{e^{ns(\frac{ns}{2})\sqrt{\mu_m}}}{\Gamma(\sqrt{\mu_m}+1)} s^\delta ds + \int_{\frac{1}{n}}^1 \frac{e^{ns}}{\sqrt{2\pi ns}} s^\delta ds \right. \right. \\
&\quad \left. \left. + \int_1^L \frac{e^{ns}}{\sqrt{2\pi ns}} s^{\delta_0-1} ds \right] + \left[\int_r^{\frac{1}{n}} \frac{e^{ns}\Gamma(\sqrt{\mu_m})2^{\sqrt{\mu_m}-1}}{(ns)\sqrt{\mu_m}} s^\delta ds + \int_{\frac{1}{n}}^1 \frac{\sqrt{\pi}e^{-ns}}{\sqrt{2ns}} s^\delta ds \right. \right. \\
&\quad \left. \left. + \int_1^L \frac{\sqrt{\pi}e^{-ns}}{\sqrt{2ns}} s^{\delta_0-1} ds \right] \right\} + C \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \frac{e^{nr}\Gamma(\sqrt{\mu_m})2^{\sqrt{\mu_m}-1}}{(nr)\sqrt{\mu_m}} \int_0^r \frac{e^{ns(\frac{ns}{2})\sqrt{\mu_m}}}{\Gamma(\sqrt{\mu_m}+1)} s^\delta ds \\
&\leq C \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \left\{ \frac{1}{e^{2nL}2^{\sqrt{\mu_m}}} \left[\int_0^{\frac{1}{n}} n^{\sqrt{\mu_m}} s^{\delta+\sqrt{\mu_m}} ds + n^{-\frac{1}{2}} \int_{\frac{1}{n}}^1 e^{ns} s^{\delta-\frac{1}{2}} ds \right. \right. \\
&\quad \left. \left. + n^{-\frac{1}{2}} \int_1^L e^{ns} s^{\delta_0-\frac{3}{2}} ds \right] + \left[\int_r^{\frac{1}{n}} \frac{e^{ns} s^\delta}{\sqrt{\mu_m}} ds + n^{-\frac{1}{2}} \int_{\frac{1}{n}}^1 \frac{e^{-ns} s^{\delta-\frac{1}{2}}}{2^{\sqrt{\mu_m}}} ds + n^{-\frac{1}{2}} \int_1^L \frac{e^{-ns} s^{\delta_0-\frac{3}{2}}}{2^{\sqrt{\mu_m}}} ds \right] \right. \\
&\quad \left. + \int_0^r \frac{e^{ns} s^\delta}{\sqrt{\mu_m}} ds \right\} \\
&\leq C \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \frac{n^{-1-\min(\frac{1}{2},\delta)}}{\sqrt{\mu_m}},
\end{aligned}$$

which yields

$$\sum_{m=1}^M \sum_{\substack{nr \leq 1 \\ 0 < r \leq 1}} |R_{mn}^{(1)}(r)| \leq C \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)}. \quad (\text{A.15})$$

Case ii. $m \leq M$, and $nr \geq 1$ with $0 < r \leq 1$.

In this case, we have

$$\begin{aligned}
|R_{mn}^{(1)}(r)| &\leq I_{\sqrt{\mu_m}}(nr) \left(\frac{K'_{\sqrt{\mu_m}}(nL)}{I'_{\sqrt{\mu_m}}(nL)} \int_0^L s I_{\sqrt{\mu_m}}(ns) |f_{Lmn}^{(1)}(s)| ds + \int_r^L s K_{\sqrt{\mu_m}}(ns) |f_{Lmn}^{(1)}(s)| ds \right) \\
&\quad + K_{\sqrt{\mu_m}}(nr) \int_0^r s I_{\sqrt{\mu_m}}(ns) |f_{Lmn}^{(1)}(s)| ds \\
&\leq C \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \frac{e^{nr}}{\sqrt{2\pi nr}} \left\{ e^{-2nL} \left[\int_0^{\frac{1}{n}} \frac{e^{ns(\frac{ns}{2})\sqrt{\mu_m}}}{\Gamma(\sqrt{\mu_m}+1)} s^\delta ds + \int_{\frac{1}{n}}^1 \frac{e^{ns}}{\sqrt{2\pi ns}} s^\delta ds \right. \right. \\
&\quad \left. \left. + \int_1^L \frac{e^{ns}}{\sqrt{2\pi ns}} s^{\delta_0-1} ds \right] + \left[\int_r^1 \frac{\sqrt{\pi}e^{-ns}}{\sqrt{2ns}} s^\delta ds + \int_1^L \frac{\sqrt{\pi}e^{-ns}}{\sqrt{2ns}} s^{\delta_0-1} ds \right] \right\} \\
&\quad + \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \frac{\sqrt{\pi}e^{-nr}}{\sqrt{2nr}} \left[\int_0^{\frac{1}{n}} \frac{e^{ns(\frac{ns}{2})\sqrt{\mu_m}}}{\Gamma(\sqrt{\mu_m}+1)} s^\delta ds + \int_{\frac{1}{n}}^r \frac{e^{ns}}{\sqrt{2\pi ns}} s^\delta ds \right]
\end{aligned}$$

$$\begin{aligned}
&\leq C\|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \left\{ \frac{n^{-1-\delta}e^{-n}}{2\sqrt{\mu_m}\Gamma(\sqrt{\mu_m}+1)} + n^{-1-\min(\frac{1}{2},\delta)} + n^{-\frac{3}{2}} + n^{-1-\min(\frac{1}{2},\delta)}e^{-nr} + n^{-\frac{3}{2}}e^{-n} \right. \\
&\quad \left. + \frac{n^{-1-\delta}}{2\sqrt{\mu_m}\Gamma(\sqrt{\mu_m}+1)} + n^{-1-\min(\frac{1}{2},\delta)} \right\} \\
&\leq C\|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} n^{-1-\min(\frac{1}{2},\delta)},
\end{aligned}$$

which derives

$$\sum_{m=1}^M \sum_{\substack{nr \geq 1 \\ 0 < r \leq 1}} |R_{mn}^{(1)}(r)| \leq C\|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)}. \quad (\text{A.16})$$

Case iii. $m \leq M$ and $1 \leq r \leq L$.

Set

$$R_{mn}^{(1)}(r) = B_{mn}^1(r) - B_{mn}^2(r) - B_{mn}^3(r), \quad (\text{A.17})$$

where

$$\begin{aligned}
B_{mn}^1(r) &= I_{\sqrt{\mu_m}}(nr) \left(\frac{K'_{\sqrt{\mu_m}}(nL)}{I'_{\sqrt{\mu_m}}(nL)} \int_0^L s I_{\sqrt{\mu_m}}(ns) f_{Lmn}^{(1)}(s) ds \right), \\
B_{mn}^2(r) &= I_{\sqrt{\mu_m}}(nr) \int_r^L s K_{\sqrt{\mu_m}}(ns) f_{Lmn}^{(1)}(s) ds \\
B_{mn}^3(r) &= K_{\sqrt{\mu_m}}(nr) \int_0^r s I_{\sqrt{\mu_m}}(ns) f_{Lmn}^{(1)}(s) ds.
\end{aligned}$$

By the same method as in Case ii and the fact of $1 \leq r \leq L$, one has

$$|B_{mn}^1(r)| \leq C\|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} n^{-\frac{3}{2}-\min(\frac{1}{2},\delta)}. \quad (\text{A.18})$$

On the other hand, we have

$$\begin{aligned}
|B_{mn}^2(r)| + |B_{mn}^3(r)| &\leq C\|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \left\{ \frac{e^{nr}}{\sqrt{2\pi nr}} \left[\int_r^L \frac{\sqrt{\pi}e^{-ns}}{\sqrt{2ns}} s^{\delta_0-1} ds \right] \right. \\
&\quad \left. + \frac{\sqrt{\pi}e^{-nr}}{\sqrt{2nr}} \left[\int_0^{\frac{1}{n}} \frac{e^{ns}(\frac{ns}{2})^{\sqrt{\mu_m}}}{\Gamma(\sqrt{\mu_m}+1)} s^{\delta} ds + \int_{\frac{1}{n}}^1 \frac{e^{ns}}{\sqrt{2\pi ns}} s^{\delta} ds + \int_1^L \frac{e^{ns}}{\sqrt{2\pi ns}} s^{\delta_0-1} ds \right] \right\} \\
&\leq C\|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \left\{ e^{nr} n^{-1} \int_r^L e^{-ns} s^{\delta_0-\frac{3}{2}} ds + e^{-nr} \left(\frac{n^{-\frac{1}{2}} \int_0^{\frac{1}{n}} s^{\delta} ds}{2\sqrt{\mu_m}\Gamma(\sqrt{\mu_m}+1)} \right. \right. \\
&\quad \left. \left. + n^{-1} \int_{\frac{1}{n}}^1 e^{ns} s^{\delta-\frac{1}{2}} ds + n^{-1} \int_1^r e^{ns} s^{\delta_0-\frac{3}{2}} ds \right) \right\} \\
&\leq C\|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \left(n^{-2} + \frac{n^{-\frac{3}{2}-\delta}}{2\sqrt{\mu_m}\Gamma(\sqrt{\mu_m}+1)} + n^{-\frac{3}{2}-\min(\frac{1}{2},\delta)} \right) \\
&\leq C\|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} n^{-\frac{3}{2}-\min(\frac{1}{2},\delta)}. \quad (\text{A.19})
\end{aligned}$$

Combining (A.18) with (A.19) yields

$$\sum_{m=1}^M \sum_{\substack{n \geq 1 \\ 0 < r \leq L}} |R_{mn}^{(1)}(r)| \leq C \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)}. \quad (\text{A.20})$$

Case iv. $m > M$ and $0 < r \leq L$.

For the notational convenience, we set

$$z(r) \equiv \frac{nr}{\sqrt{\mu_m}}, \quad \eta(r) \equiv \sqrt{1+z^2(r)} + \ln \frac{z(r)}{1+\sqrt{1+z^2(r)}}, \quad F(r) \equiv e^{-\sqrt{\mu_m}\eta(r)}, \quad \tilde{F}(r) = -\frac{1}{F(r)}. \quad (\text{A.21})$$

It is easy to know that $F(r)$ is decreasing and $\tilde{F}(r)$ is increasing with respect to r .

By (viii) in Lemma 2.3, we obtain

$$I_{\sqrt{\mu_m}}(nr) = I_{\sqrt{\mu_m}}\left(\sqrt{\mu_m} \frac{nr}{\sqrt{\mu_m}}\right) \sim \frac{1}{\sqrt{2\pi\sqrt{\mu_m}}} \frac{\tilde{F}(r)}{(1+z^2(r))^{\frac{1}{4}}}; \quad (\text{A.22})$$

$$K_{\sqrt{\mu_m}}(nr) \sim \sqrt{\frac{\pi}{2\sqrt{\mu_m}}} \frac{F(r)}{(1+z^2(r))^{\frac{1}{4}}}; \quad (\text{A.23})$$

$$I'_{\sqrt{\mu_m}}(nr) \sim \frac{1}{\sqrt{2\pi\sqrt{\mu_m}}} \frac{(1+z^2(r))^{\frac{1}{4}}}{z(r)} \tilde{F}(r); \quad (\text{A.24})$$

$$K'_{\sqrt{\mu_m}}(nr) \sim -\sqrt{\frac{\pi}{2\sqrt{\mu_m}}} \frac{(1+z^2(r))^{\frac{1}{4}}}{z(r)} F(r). \quad (\text{A.25})$$

By (A.17), we have

$$|R_{mn}^{(1)}(r)| \leq |B_{mn}^1| + |B_{mn}^2| + |B_{mn}^3|. \quad (\text{A.26})$$

Next we deal with each B_{mn}^i ($i = 1, 2, 3$) in (A.26) separately.

(A) Estimation of B_{mn}^1

By the inequalities in (4.35), (4.39) and (A.22)-(A.25), we obtain that

$$\begin{aligned} |B_{mn}^1| &\leq I_{\sqrt{\mu_m}}(nr) \left| \frac{K'_{\sqrt{\mu_m}}(nL)}{I'_{\sqrt{\mu_m}}(nL)} \right| \int_0^L s I_{\sqrt{\mu_m}}(ns) |f_{Lmn}^{(1)}(s)| ds \\ &\leq C \frac{F(L)}{\tilde{F}(L)} \frac{\tilde{F}(r)}{\sqrt{\mu_m}} \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)} n^{-1} \left\{ \int_0^1 \frac{\tilde{F}(s)s^{\delta-1}}{(1+z^2(s))^{\frac{1}{4}}} ds + \int_1^L \frac{\tilde{F}(s)s^{\delta_0-2}}{(1+z^2(s))^{\frac{1}{4}}} ds \right\} \\ &\leq C \frac{F(L)}{\tilde{F}(L)} \frac{\tilde{F}(r)}{\sqrt{\mu_m}} \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)} n^{-1} \left\{ \int_0^1 \tilde{F}'(s) \frac{s^\delta}{(1+z^2(s))^{\frac{1}{4}}} \frac{z(s)}{ns} \frac{1}{\sqrt{1+z^2(s)}} ds \right. \\ &\quad \left. + \int_1^L \tilde{F}'(s) \frac{s^{\delta_0-1}}{(1+z^2(s))^{\frac{1}{4}}} \frac{z(s)}{ns} \frac{1}{\sqrt{1+z^2(s)}} ds \right\} \end{aligned}$$

$$\begin{aligned}
&\leq C \frac{F(L)}{\tilde{F}(L)} \frac{\tilde{F}(r)}{\sqrt{\mu_m}} \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)} n^{-1} \left\{ \int_0^1 \tilde{F}'(s) \left(\frac{s}{z(s)} \right)^\delta \frac{\mu_m^{-\frac{1}{2}}}{(1+z^2(s))^{\frac{3}{4}-\frac{\delta}{2}}} ds \right. \\
&\quad \left. + \int_1^L \tilde{F}'(s) \left(\frac{s}{z(s)} \right)^{\delta_0} \frac{\mu_m^{-\frac{1}{2}}}{(1+z^2(s))^{\frac{3}{4}-\frac{\delta_0}{2}}} ds \right\} \\
&\leq C \frac{F(L)\tilde{F}(r)}{\tilde{F}(L)} \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)} \left(\tilde{F}(1)n^{-1-\delta}\mu_m^{-1+\frac{\delta}{2}} + \tilde{F}(L)n^{-1-\delta_0}\mu_m^{-1+\frac{\delta_0}{2}} \right) \\
&\leq C \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)} \left(n^{-1-\delta}\mu_m^{-1+\frac{\delta}{2}} + n^{-1-\delta_0}\mu_m^{-1+\frac{\delta_0}{2}} \right). \tag{A.27}
\end{aligned}$$

(B) Estimation of B_{mn}^2

We will treat B_{mn}^2 in two cases of $0 < r \leq 1$ and $r \geq 1$.

Case (a) $0 < r \leq 1$.

$$\begin{aligned}
|B_{mn}^2| &\leq I_{\sqrt{\mu_m}}(nr) \left\{ \int_r^1 s K_{\sqrt{\mu_m}}(ns) |f_{Lmn}^{(1)}(s)| ds + \int_1^L s K_{\sqrt{\mu_m}}(ns) |f_{Lmn}^{(1)}(s)| ds \right\} \\
&\leq C \frac{1}{n\sqrt{\mu_m}} \frac{\tilde{F}(r)}{(1+z^2(r))^{\frac{1}{4}}} \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)} \left\{ \int_r^1 \frac{F(s)s^{\delta-1}}{(1+z^2(s))^{\frac{1}{4}}} ds + \int_1^L \frac{F(s)s^{\delta_0-2}}{(1+z^2(s))^{\frac{1}{4}}} ds \right\} \\
&\leq C \frac{1}{n\sqrt{\mu_m}} \frac{\tilde{F}(r)}{(1+z^2(r))^{\frac{1}{4}}} \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)} \left\{ \int_r^1 \left(-F'(s) \right) \left(\frac{s}{z(s)} \right)^\delta \frac{\mu_m^{-\frac{1}{2}}}{(1+z^2(s))^{\frac{3}{4}-\frac{\delta}{2}}} ds \right. \\
&\quad \left. + \int_1^L \left(-F'(s) \right) \left(\frac{s}{z(s)} \right)^{\delta_0} \frac{\mu_m^{-\frac{1}{2}}}{(1+z^2(s))^{\frac{3}{4}-\frac{\delta_0}{2}}} ds \right\} \\
&\leq C \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)} \tilde{F}(r) \left(F(r)n^{-1-\delta}\mu_m^{-1+\frac{\delta}{2}} + F(1)n^{-1-\delta_0}\mu_m^{-1+\frac{\delta_0}{2}} \right) \\
&\leq C \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)} \left(n^{-1-\delta}\mu_m^{-1+\frac{\delta}{2}} + n^{-1-\delta_0}\mu_m^{-1+\frac{\delta_0}{2}} \right). \tag{A.28}
\end{aligned}$$

Case (b) $1 \leq r \leq L$

As in case (a), we can arrive at

$$\begin{aligned}
|B_{mn}^2| &\leq I_{\sqrt{\mu_m}}(nr) \int_r^L s K_{\sqrt{\mu_m}}(ns) |f_{Lmn}^{(1)}(s)| ds \\
&\leq C \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)} n^{-1-\delta_0} \mu_m^{-1+\frac{\delta_0}{2}}. \tag{A.29}
\end{aligned}$$

(C) Estimation of B_{mn}^3

As in (B) above, we also treat B_{mn}^3 in two cases of $0 < r \leq 1$ and $r \geq 1$ separately.

Case (a) $0 < r \leq 1$

$$\begin{aligned}
|B_{mn}^3| &\leq K_{\sqrt{\mu_m}}(nr) \int_0^r s I_{\sqrt{\mu_m}}(ns) |f_{Lmn}^{(1)}(s)| ds \\
&\leq C \sqrt{\frac{\pi}{2\sqrt{\mu_m}}} \frac{F(r)}{(1+z^2(r))^{\frac{1}{4}}} \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)} n^{-1} \left\{ \int_0^r \frac{1}{\sqrt{2\pi\sqrt{\mu_m}}} \frac{\tilde{F}(r)}{(1+z^2(r))^{\frac{1}{4}}} s^{\delta-1} ds \right\} \\
&\leq C \frac{1}{n\sqrt{\mu_m}} F(r) \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)} \left\{ \int_0^r \tilde{F}'(s) \frac{s^\delta}{(1+z^2(s))^{\frac{1}{4}}} \frac{z(s)}{ns} \frac{1}{\sqrt{1+z^2(s)}} ds \right\} \\
&\leq C \frac{1}{n\sqrt{\mu_m}} F(r) \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)} \left\{ \int_0^r \tilde{F}'(s) \left(\frac{s}{z(s)} \right)^\delta \frac{\mu_m^{-\frac{1}{2}}}{(1+z^2(s))^{\frac{3}{4}-\frac{\delta}{2}}} ds \right\} \\
&\leq CF(r) \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)} \tilde{F}(r) n^{-1-\delta} \mu_m^{-1+\frac{\delta}{2}} \\
&\leq C \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)} n^{-1-\delta} \mu_m^{-1+\frac{\delta}{2}}.
\end{aligned} \tag{A.30}$$

Case (b) $1 \leq r \leq L$

In this case, we have

$$\begin{aligned}
|B_{mn}^3| &\leq K_{\sqrt{\mu_m}}(nr) \left\{ \int_0^1 s I_{\sqrt{\mu_m}}(ns) |f_{Lmn}^{(1)}(s)| ds + \int_0^1 s I_{\sqrt{\mu_m}}(ns) |f_{Lmn}^{(1)}(s)| ds \right\} \\
&\leq C \frac{1}{n\sqrt{\mu_m}} F(r) \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)} \left\{ \int_0^1 \tilde{F}'(s) \left(\frac{s}{z(s)} \right)^\delta \frac{\mu_m^{-\frac{1}{2}}}{(1+z^2(s))^{\frac{3}{4}-\frac{\delta}{2}}} ds \right. \\
&\quad \left. + \int_1^r \tilde{F}'(s) \left(\frac{s}{z(s)} \right)^{\delta_0} \frac{\mu_m^{-\frac{1}{2}}}{(1+z^2(s))^{\frac{3}{4}-\frac{\delta_0}{2}}} ds \right\} \\
&\leq C \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)} F(r) \left(\tilde{F}(1) n^{-1-\delta} \mu_m^{-1+\frac{\delta}{2}} + \tilde{F}(r) n^{-1-\delta_0} \mu_m^{-1+\frac{\delta_0}{2}} \right) \\
&\leq C \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)} \left(n^{-1-\delta} \mu_m^{-1+\frac{\delta}{2}} + n^{-1-\delta_0} \mu_m^{-1+\frac{\delta_0}{2}} \right).
\end{aligned} \tag{A.31}$$

Substituting (A.27)-(A.31) into (A.26) yields

$$\sum_{m=M}^{\infty} \sum_{n=1}^{\infty} |R_{mn}^{(1)}(r)| \leq C \|f\|_{1,\alpha;Q}^{(1-\delta,2-\delta_0)}. \tag{A.32}$$

By (A.15)-(A.16), (A.20) and (A.32), we complete the proof of Lemma A.4. \square

Lemma A.5. For the term $\sum_{n=1}^{\infty} n^2 |R_{0n}^{(1)}(r)|$ defined in (4.46), we have

$$\sum_{n=1}^{\infty} n^2 |R_{0n}^{(1)}(r)| \leq \begin{cases} C \|f\|_{3,\alpha;Q}^{(1-\delta,2-\delta_0)} r^{\delta-\frac{5}{2}}, & 0 < r \leq 1, \\ C \|f\|_{3,\alpha;Q}^{(1-\delta,2-\delta_0)}, & 1 < r \leq L, \end{cases} \tag{A.33}$$

where $C > 0$ is independent of L .

Proof. As in (A.5), we write $R_{0n}^{(1)}(r) = B_1^n - B_2^n$. First, we estimate $\sum_{n=1}^{\infty} n^2 |B_1^n(r)|$. It follows from (iii)-(vii) of Lemma 2.3, the inequalities in (4.34) and (4.38), and the fact of $L \geq 4$ that

$$\begin{aligned}
\sum_{n=1}^{\infty} n^2 |B_1^n(r)| &\leq \sum_{n=1}^{\infty} n^2 \frac{\sqrt{\frac{\pi}{2nL}} e^{-nL}}{\frac{e^{nL}}{\sqrt{2\pi nL}}} e^{nr} \left(\int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^1 + \int_1^L \right) s I_0(ns) |f_{L0n}^{(1)}(s)| ds \\
&\leq C \sum_{n=1}^{\infty} n^2 e^{nr-2nL} \left\{ \int_0^{\frac{1}{n}} e^{ns} s^{\delta} ds \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} + \left(\int_{\frac{1}{n}}^1 e^{ns} s^{\delta-3} ds \right. \right. \\
&\quad \left. \left. + \int_1^L e^{ns} s^{\delta_0-4} ds \right) n^{-3} \|f\|_{3,\alpha;Q}^{(1-\delta,2-\delta_0)} \right\} \\
&\leq C \sum_{n=1}^{\infty} \left\{ n^{1-\delta} e^{-nL} \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} + (n^{1-\delta} e^{n(1-L)} + n^{-2}) \|f\|_{3,\alpha;Q}^{(1-\delta,2-\delta_0)} \right\} \\
&\leq C \|f\|_{3,\alpha;Q}^{(1-\delta,2-\delta_0)} \sum_{n=1}^{\infty} (n^{1-\delta} e^{-2n} + n^{1-\delta} e^{-n} + n^{-2}) \\
&\leq C \|f\|_{3,\alpha;Q}^{(1-\delta,2-\delta_0)}. \tag{A.34}
\end{aligned}$$

To estimate $\sum_{n=1}^{\infty} n^2 |B_2^n(r)|$, we will divide this procedure into the following cases:

Case i. $0 < r \leq 1$

In this case, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} n^2 |B_2^n(r)| &\leq \sum_{n=1}^{\infty} n^2 \left\{ I_0(nr) \left(\int_r^1 K_0(ns) |f_{L0n}^{(1)}(s)| ds + \int_1^L s K_0(ns) |f_{L0n}^{(1)}(s)| ds \right) \right. \\
&\quad \left. + K_0(nr) \left(\int_0^{\frac{r}{2}} s I_0(ns) |f_{L0n}^{(1)}(s)| ds + \int_{\frac{r}{2}}^r s I_0(ns) |f_{L0n}^{(1)}(s)| ds \right) \right\} \\
&\leq \sum_{n=1}^{\infty} C n^2 \left\{ e^{nr} n^{-2} \|f\|_{2,\alpha;Q}^{(1-\delta,2-\delta_0)} \left(\int_r^1 \frac{e^{-ns}}{\sqrt{2ns}} s^{\delta-2} ds + \int_1^L \frac{e^{-ns}}{\sqrt{2ns}} s^{\delta_0-3} ds \right) \right. \\
&\quad \left. + \frac{e^{-nr}}{\sqrt{2nr}} \left(\|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \int_0^{\frac{r}{2}} e^{ns} s^{\delta} ds + \|f\|_{2,\alpha;Q}^{(1-\delta,2-\delta_0)} n^{-2} \int_{\frac{r}{2}}^r e^{ns} s^{\delta-2} ds \right) \right\} \\
&\leq C \|f\|_{2,\alpha;Q}^{(1-\delta,2-\delta_0)} \sum_{n=1}^{\infty} \left\{ e^{nr} \left(n^{-\frac{1}{2}} \int_r^1 e^{-ns} s^{\delta-\frac{5}{2}} ds + n^{-\frac{1}{2}} \int_1^L e^{-ns} s^{\delta_0-\frac{7}{2}} ds \right) \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{e^{-nr}}{\sqrt{2nr}} \left(n^2 \int_0^{\frac{r}{2}} e^{ns} s^\delta ds + \int_{\frac{r}{2}}^r e^{ns} s^{\delta-2} ds \right) \Big\} \\
& \leq C \|f\|_{2,\alpha;Q}^{(1-\delta,2-\delta_0)} \sum_{n=1}^{\infty} \left(n^{-\frac{3}{2}} r^{\delta-\frac{5}{2}} + e^{-\frac{nr}{2}} n^{\frac{1}{2}} r^{\delta-\frac{1}{2}} + r^{\delta-\frac{5}{2}} n^{-\frac{3}{2}} \right) \\
& \leq C \|f\|_{2,\alpha;Q}^{(1-\delta,2-\delta_0)} r^{\delta-\frac{5}{2}}. \tag{A.35}
\end{aligned}$$

Case ii. $1 \leq r \leq L$

At this time, we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} n^2 |B_2^n(r)| & \leq \sum_{n=1}^{\infty} n^2 \left\{ e^{nr} \int_r^L K_0(ns) |f_{L0n}^{(1)}(s)| ds \right. \\
& \quad \left. + K_0(nr) \left(\int_0^{\frac{1}{2}} s I_0(ns) |f_{L0n}^{(1)}(s)| ds + \int_{\frac{1}{2}}^1 s I_0(ns) |f_{L0n}^{(1)}(s)| ds + \int_1^r s I_0(ns) |f_{L0n}^{(1)}(s)| ds \right) \right\} \\
& \leq C \sum_{n=1}^{\infty} \left\{ e^{nr} \|f\|_{2,\alpha;Q}^{(1-\delta,2-\delta_0)} n^{-2} \int_r^L \frac{e^{-ns}}{\sqrt{2ns}} s^{\delta_0-3} ds \right. \\
& \quad \left. + \frac{e^{-nr}}{\sqrt{2nr}} \left(\|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \int_0^{\frac{1}{2}} e^{ns} s^\delta + \|f\|_{2,\alpha}^{(1-\delta,2-\delta_0)} n^{-2} \left[\int_{\frac{1}{2}}^1 e^{ns} s^{\delta-2} ds + \int_1^r e^{ns} s^{\delta_0-3} ds \right] \right) \right\} \\
& \leq C \|f\|_{2,\alpha;Q}^{(1-\delta,2-\delta_0)} \sum_{n=1}^{\infty} \left\{ e^{nr} (n^{-\frac{3}{2}} e^{-nr}) + e^{-nr} \left(n^{\frac{1}{2}} e^{\frac{n}{2}} + n^{-\frac{3}{2}} e^n + n^{-\frac{3}{2}} e^{nr} \right) \right\} \\
& \leq C \|f\|_{2,\alpha;Q}^{(1-\delta,2-\delta_0)} \sum_{n=1}^{\infty} (n^{\frac{1}{2}} e^{-\frac{n}{2}} + n^{-\frac{3}{2}}) \\
& \leq C \|f\|_{2,\alpha;Q}^{(1-\delta,2-\delta_0)}. \tag{A.36}
\end{aligned}$$

Combining (A.34)-(A.36) yields (A.33). \square

Lemma A.6. For the term $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^2 |R_{mn}^{(1)}(r)|$ defined in (4.46), we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^2 |R_{mn}^{(1)}(r)| \leq \begin{cases} C \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)} r^{\min\{\delta, \frac{1}{2}\} - \frac{5}{2}}, & r \leq 1, \\ C \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)}, & 1 < r \leq L, \end{cases} \tag{A.37}$$

where $C > 0$ is independent of L .

Proof. To prove (A.37), we will divide this procedure into the following four cases. As in (A.26), one has $|R_{mn}^{(1)}(r)| \leq B_{mn}^1 + B_{mn}^2 + B_{mn}^3$.

Case i. $m \leq M$, and $n \leq \frac{1}{r}$ with $0 < r \leq 1$

At this time, we can choose a positive integer N such that $Nr \leq 1$ holds. And by

(iv)-(vii) of Lemma 2.3 and the inequalities in (4.35) and (4.39), we have

$$\begin{aligned}
n^2 |R_{mn}^{(1)}(r)| &\leq n^2 I_{\sqrt{\mu_m}}(nr) \left(\frac{K'_{\sqrt{\mu_m}}(nL)}{I'_{\sqrt{\mu_m}}(nL)} \int_0^L s I_{\sqrt{\mu_m}}(ns) |f_{Lmn}^{(1)}(s)| ds \right. \\
&\quad \left. + \int_r^L s K_{\sqrt{\mu_m}}(ns) |f_{Lmn}^{(1)}(s)| ds \right) + n^2 K_{\sqrt{\mu_m}}(nr) \int_0^r s I_{\sqrt{\mu_m}}(ns) |f_{Lmn}^{(1)}(s)| ds \\
&\leq C \|f\|_{2,\alpha;Q}^{(1-\delta,2-\delta_0)} \frac{e^{nr} \left(\frac{nr}{2}\right)^{\sqrt{\mu_m}}}{\Gamma(\sqrt{\mu_m}+1)} n^2 \left\{ e^{-2nL} \left[\int_0^{\frac{1}{n}} \frac{e^{ns} \left(\frac{ns}{2}\right)^{\sqrt{\mu_m}} s^\delta}{\Gamma(\sqrt{\mu_m}+1)} ds + \int_{\frac{1}{n}}^1 \frac{e^{ns} s^\delta}{\sqrt{2\pi ns}} ds \right. \right. \\
&\quad \left. \left. + \frac{1}{n^2} \int_1^L \frac{e^{ns} s^{\delta_0-3}}{\sqrt{2\pi ns}} ds \right] + n^{-2} \left[\int_r^{Nr} \frac{e^{ns} \Gamma(\sqrt{\mu_m}) 2^{\sqrt{\mu_m}-1}}{(ns)^{\sqrt{\mu_m}}} s^{\delta-2} ds + \int_{Nr}^1 \frac{\sqrt{\pi} e^{-ns}}{\sqrt{2ns}} s^{\delta-2} ds \right. \right. \\
&\quad \left. \left. + \int_1^L \frac{\sqrt{\pi} e^{-ns}}{\sqrt{2ns}} s^{\delta_0-3} ds \right] \right\} + \|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \frac{e^{nr} \Gamma(\sqrt{\mu_m}) 2^{\sqrt{\mu_m}-1}}{(nr)^{\sqrt{\mu_m}}} \int_0^r \frac{e^{ns} \left(\frac{ns}{2}\right)^{\sqrt{\mu_m}}}{\Gamma(\sqrt{\mu_m}+1)} s^\delta ds \\
&\leq C \|f\|_{2,\alpha;Q}^{(1-\delta,2-\delta_0)} n^2 \left\{ \frac{1}{e^{2nL} 2^{\sqrt{\mu_m}} \Gamma(\sqrt{\mu_m}+1)} \left[\int_0^{\frac{1}{n}} s^\delta ds + n^{-\frac{1}{2}} \int_{\frac{1}{n}}^1 e^{ns} s^{\delta-\frac{1}{2}} ds \right. \right. \\
&\quad \left. \left. + n^{-\frac{3}{2}} \int_1^L e^{ns} s^{\delta_0-\frac{7}{2}} ds \right] + n^{-2} \left[\int_r^{Nr} \frac{e^{ns} s^{\delta-2}}{\sqrt{\mu_m}} ds + \int_{Nr}^1 \frac{e^{-ns} s^{\delta-2}}{2^{\sqrt{\mu_m}}} ds \right. \right. \\
&\quad \left. \left. + n^{-\frac{1}{2}} \int_1^L \frac{e^{-ns} s^{\delta_0-\frac{7}{2}}}{2^{\sqrt{\mu_m}}} ds \right] + \int_0^r \frac{e^{ns} s^\delta}{\sqrt{\mu_m}} ds \right\} \\
&\leq C \|f\|_{2,\alpha;Q}^{(1-\delta,2-\delta_0)} \left(\frac{n^{1-\min(\frac{1}{2},\delta)} e^{-n}}{2^{\sqrt{\mu_m}}} + \frac{n^{-\frac{3}{2}} e^{-n}}{2^{\sqrt{\mu_m}}} + \frac{n^{-1}}{\sqrt{\mu_m}} r^{\delta-2} + \frac{n^{-1} e^{-\frac{n}{2}}}{2^{\sqrt{\mu_m}}} \right).
\end{aligned}$$

Thus we get

$$\sum_{m=1}^M \sum_{\substack{n \leq \frac{1}{r} \\ 0 < r \leq 1}} n^2 |R_{mn}^{(1)}(r)| \leq C \|f\|_{2,\alpha;Q}^{(1-\delta,2-\delta_0)} r^{\delta-2}. \quad (\text{A.38})$$

Case ii. $m \leq M$, and $n \geq \frac{1}{r}$ with $0 < r \leq 1$

In this case, we get

$$\begin{aligned}
n^2 |R_{mn}^{(1)}(r)| &\leq C n^2 \|f\|_{2,\alpha;Q}^{(1-\delta,2-\delta_0)} \frac{e^{nr}}{\sqrt{2\pi nr}} \left\{ e^{-2nL} \left[\int_0^{\frac{1}{n}} \frac{e^{ns} \left(\frac{ns}{2}\right)^{\sqrt{\mu_m}}}{\Gamma(\sqrt{\mu_m}+1)} s^\delta ds + \int_{\frac{1}{n}}^1 \frac{e^{ns}}{\sqrt{2\pi ns}} s^\delta ds \right. \right. \\
&\quad \left. \left. + n^{-2} \int_1^L \frac{e^{ns}}{\sqrt{2\pi ns}} s^{\delta_0-3} ds \right] + n^{-2} \left[\int_r^1 \frac{\sqrt{\pi} e^{-ns}}{\sqrt{2ns}} s^{\delta-2} ds + \int_1^L \frac{\sqrt{\pi} e^{-ns}}{\sqrt{2ns}} s^{\delta_0-3} ds \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& + n^2 \frac{\sqrt{\pi} e^{-nr}}{\sqrt{2nr}} \left[\|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \int_0^{\frac{r}{2}} \frac{e^{ns}}{\sqrt{2\pi ns}} s^\delta ds + n^{-2} \|f\|_{2,\alpha}^{(1-\delta,2-\delta_0)} \int_{\frac{r}{2}}^r \frac{e^{ns}}{\sqrt{2\pi ns}} s^{\delta-2} ds \right] \\
& \leq C \|f\|_{2,\alpha;Q}^{(1-\delta,2-\delta_0)} n^2 \left\{ e^{nr-2nL} \left[\frac{\int_0^{\frac{1}{n}} s^\delta ds}{2\sqrt{\mu_m} \Gamma(\sqrt{\mu_m} + 1)} + n^{-\frac{1}{2}} \int_{\frac{1}{n}}^1 e^{ns} s^{\delta-\frac{1}{2}} ds + n^{-\frac{5}{2}} \int_1^L e^{ns} s^{\delta_0-\frac{7}{2}} ds \right] \right. \\
& \quad + e^{nr} n^{-\frac{5}{2}} \left[\int_r^1 e^{-ns} s^{\delta-\frac{5}{2}} ds + \int_1^L e^{-ns} s^{\delta_0-\frac{7}{2}} ds \right] + e^{-nr} n^{-\frac{1}{2}} \left[\int_0^{\frac{r}{2}} e^{ns} s^{\delta-\frac{1}{2}} ds \right. \\
& \quad \left. \left. + n^{-2} \int_{\frac{r}{2}}^r e^{ns} s^{\delta-\frac{5}{2}} ds \right] \right\} \\
& \leq C \|f\|_{2,\alpha;Q}^{(1-\delta,2-\delta_0)} \left(n^{1-\min(\frac{1}{2},\delta)} e^{-n} + n^{-\frac{3}{2}} + n^{-\frac{3}{2}} r^{\delta-\frac{5}{2}} + n^{\frac{3}{2}} e^{-\frac{nr}{2}} r^{\delta+\frac{1}{2}} \right).
\end{aligned}$$

This, together with $\sum_{n \geq \frac{1}{r}} n^{\frac{3}{2}} e^{-\frac{nr}{2}} r^{\delta+\frac{1}{2}} \leq C r^{\delta-2}$, yields

$$\sum_{m=1}^M \sum_{\substack{n \geq \frac{1}{r} \\ 0 < r \leq 1}} n^2 |R_{mn}^{(1)}(r)| \leq C \|f\|_{2,\alpha;Q}^{(1-\delta,2-\delta_0)} r^{\delta-\frac{5}{2}}. \quad (\text{A.39})$$

Case iii. $m \leq M$ and $1 \leq r \leq L$

As in Case (b), one has

$$n^2 |B_{mn}^1| \leq C \|f\|_{2,\alpha;Q}^{(1-\delta,2-\delta_0)} \left(\frac{n^{1-\delta} e^{-n}}{2\sqrt{\mu_m} \Gamma(\sqrt{\mu_m} + 1)} + n^{1-\min(\frac{1}{2},\delta)} e^{-n} + n^{-\frac{3}{2}} \right).$$

In addition,

$$\begin{aligned}
& n^2 (|B_{mn}^2| + |B_{mn}^3|) \\
& \leq C n^2 \left\{ \frac{e^{nr}}{\sqrt{2\pi nr}} \|f\|_{2,\alpha;Q}^{(1-\delta,2-\delta_0)} n^{-2} \left[\int_r^L \frac{\sqrt{\pi} e^{-ns}}{\sqrt{2ns}} s^{\delta_0-3} ds \right] \right. \\
& \quad + \frac{\sqrt{\pi} e^{-nr}}{\sqrt{2nr}} \left[\|f\|_{0,\alpha;Q}^{(1-\delta,2-\delta_0)} \int_0^{\frac{1}{2}} \frac{e^{ns}}{\sqrt{2\pi ns}} s^\delta ds \right. \\
& \quad \left. \left. + \|f\|_{2,\alpha;Q}^{(1-\delta,2-\delta_0)} n^{-2} \left(\int_{\frac{1}{2}}^1 \frac{e^{ns}}{\sqrt{2\pi ns}} s^{\delta-2} ds + \int_1^L \frac{e^{ns}}{\sqrt{2\pi ns}} s^{\delta_0-3} ds \right) \right] \right\} \\
& \leq C \|f\|_{2,\alpha;Q}^{(1-\delta,2-\delta_0)} \left\{ e^{nr} n^{-1} \int_r^L e^{-ns} s^{\delta_0-\frac{7}{2}} ds \right. \\
& \quad \left. + e^{-nr} \left(n \int_0^{\frac{1}{2}} e^{ns} s^{\delta-\frac{1}{2}} ds + n^{-1} \int_{\frac{1}{2}}^1 e^{ns} s^{\delta-\frac{5}{2}} ds + n^{-1} \int_1^r e^{ns} s^{\delta_0-\frac{7}{2}} ds \right) \right\} \\
& \leq C \|f\|_{2,\alpha;Q}^{(1-\delta,2-\delta_0)} (n^{-2} + n e^{-n}).
\end{aligned}$$

Hence, we get

$$\sum_{m=1}^M \sum_{\substack{n \geq 1 \\ 1 \leq r \leq 1}} n^2 |R_{mn}^{(1)}(r)| \leq C \|f\|_{2,\alpha;Q}^{(1-\delta,2-\delta_0)}. \quad (\text{A.40})$$

Case iv. $m > M$ and $0 < r \leq L$

Assume that $z(r), \eta(r), F(r), \tilde{F}(r)$ are the functions defined in (A.21). Following (ix) in Lemma 2.3, it is easy to know that $F(r_2)\tilde{F}(r_1) \leq e^{-n(r_2-r_1)}$ holds for $r_1 \leq r_2$. This, together with (A.22)-(A.25) and a direct computation, yields

$$\begin{aligned}
n^2|B_{mn}^1| &\leq C \frac{F(L)}{\tilde{F}(L)} \frac{\tilde{F}(r)n^2}{\sqrt{\mu_m}} \left\{ \|f\|_{2,\alpha;Q}^{(1-\delta,2-\delta_0)} \mu_m^{-1} \int_0^1 \frac{\tilde{F}(s)s^\delta}{(1+z^2(s))^{\frac{1}{4}}} ds \right. \\
&\quad \left. + \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)} \mu_m^{-1} n^{-2} \int_1^L \frac{\tilde{F}(s)s^{\delta_0-3}}{(1+z^2(s))^{\frac{1}{4}}} ds \right\} \\
&\leq C \frac{F(L)\tilde{F}(r)}{\tilde{F}(L)} \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)} n^2 \mu_m^{-\frac{3}{2}} \left\{ \int_0^1 \tilde{F}'(s) \frac{s^\delta}{(1+z^2(s))^{\frac{1}{4}}} \frac{z(s)}{n\sqrt{1+z^2(s)}} ds \right. \\
&\quad \left. + n^{-2} \int_1^L \tilde{F}'(s) \frac{s^{\delta_0-3}}{(1+z^2(s))^{\frac{1}{4}}} \frac{z(s)}{n\sqrt{1+z^2(s)}} ds \right\} \\
&\leq C \frac{F(L)\tilde{F}(r)}{\tilde{F}(L)} \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)} n^2 \mu_m^{-\frac{3}{2}} \left\{ n^{-1} \int_0^1 \tilde{F}'(s) \frac{z(s)}{\sqrt{1+z^2(s)}} ds \right. \\
&\quad \left. + n^{-3} \int_1^L \tilde{F}'(s) \frac{z(s)}{\sqrt{1+z^2(s)}} \left(\frac{s}{z(s)} \right)^{\frac{1}{2}} s^{\delta_0-\frac{5}{2}} ds \right\} \\
&\leq C \frac{\tilde{F}(r)}{\tilde{F}(L)} \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)} \left(n \mu_m^{-\frac{3}{2}} F(L) \tilde{F}(1) + n^{-\frac{3}{2}} \mu_m^{-\frac{5}{4}} F(L) \tilde{F}(L) \right) \\
&\leq C \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)} \left(n \mu_m^{-\frac{3}{2}} e^{-n(L-1)} + n^{-\frac{3}{2}} \mu_m^{-\frac{5}{4}} \right) \\
&\leq C \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)} \left(n e^{-n} \mu_m^{-\frac{3}{2}} + n^{-\frac{3}{2}} \mu_m^{-\frac{5}{4}} \right). \tag{A.41}
\end{aligned}$$

In addition, we obtain for $r \leq 1$

$$\begin{aligned}
n^2|B_{mn}^2| &\leq C \frac{n^2}{\sqrt{\mu_m}} \frac{\tilde{F}(r)}{(1+z^2(r))^{\frac{1}{4}}} \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)} \left\{ n^{-2} \mu_m^{-1} \int_r^1 \frac{F(s)s^{\delta-2}}{(1+z^2(s))^{\frac{1}{4}}} ds \right. \\
&\quad \left. + n^{-2} \mu_m^{-1} \int_1^L \frac{F(s)s^{\delta_0-3}}{(1+z^2(s))^{\frac{1}{4}}} ds \right\} \\
&\leq C \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)} \tilde{F}(r) n^{-1} \mu_m^{-\frac{3}{2}} \left\{ \int_r^1 \left(-F'(s) \right) \frac{z(s)}{\sqrt{1+z^2(s)}} \left(\frac{1}{z(s)} \right)^{\frac{1}{2}} s^{\delta-2} ds \right. \\
&\quad \left. + \int_1^L \left(-F'(s) \right) \frac{z(s)}{\sqrt{1+z^2(s)}} \left(\frac{1}{z(r)} \right)^{\frac{1}{2}} s^{\delta_0-3} ds \right\} \\
&\leq C \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)} n^{-\frac{3}{2}} \mu_m^{-\frac{5}{4}} \left(\tilde{F}(r) F(r) r^{\delta-\frac{5}{2}} + \tilde{F}(r) F(1) \right) \\
&\leq C \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)} n^{-\frac{3}{2}} \mu_m^{-\frac{5}{4}} r^{\delta-\frac{5}{2}}, \tag{A.42}
\end{aligned}$$

and for $r \geq 1$,

$$\begin{aligned} n^2 |B_{mn}^2| &\leq n^2 I_{\sqrt{\mu_m}}(nr) \int_r^L s K_{\sqrt{\mu_m}}(ns) |f_{Lmn}^{(1)}(s)| ds \\ &\leq C \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)} n^{-\frac{3}{2}} \mu_m^{-\frac{5}{4}}. \end{aligned} \quad (\text{A.43})$$

Finally, we estimate $n^2 |B_{mn}^3|$. For $0 < r \leq 1$, we have

$$\begin{aligned} n^2 |B_{mn}^3| &\leq n^2 K_{\sqrt{\mu_m}}(nr) \left(\int_0^{\frac{r}{2}} s I_{\sqrt{\mu_m}}(ns) |f_{Lmn}^{(1)}(s)| ds + \int_{\frac{r}{2}}^r s I_{\sqrt{\mu_m}}(ns) |f_{Lmn}^{(1)}(s)| ds \right) \\ &\leq C \frac{n^2}{\sqrt{\mu_m}} \frac{F(r)}{(1+z^2(r))^{\frac{1}{4}}} \left\{ \|f\|_{2,\alpha;Q}^{(1-\delta,2-\delta_0)} \mu_m^{-1} \int_0^{\frac{r}{2}} \frac{\tilde{F}(r)}{(1+z^2(r))^{\frac{1}{4}}} s^\delta ds \right. \\ &\quad \left. + \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)} n^{-2} \mu_m^{-1} \int_{\frac{r}{2}}^r \frac{\tilde{F}(r)}{(1+z^2(r))^{\frac{1}{4}}} s^{\delta-2} ds \right\} \\ &\leq C \frac{n^2}{\sqrt{\mu_m}} F(r) \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)} \left\{ n^{-1} \mu_m^{-1} \int_0^{\frac{r}{2}} \tilde{F}'(s) \frac{s^\delta}{(1+z^2(s))^{\frac{1}{4}}} \frac{z(s)}{\sqrt{1+z^2(s)}} ds \right. \\ &\quad \left. + n^{-3} \mu_m^{-1} \int_{\frac{r}{2}}^r \tilde{F}'(s) \frac{s^{\delta-2}}{(1+z^2(s))^{\frac{1}{4}}} \frac{z(s)}{\sqrt{1+z^2(s)}} ds \right\} \\ &\leq C \frac{n^2}{\sqrt{\mu_m}} F(r) \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)} \left(n^{-1} \mu_m^{-1} \int_0^{\frac{r}{2}} \tilde{F}'(s) ds + n^{-3} \mu_m^{-1} \int_{\frac{r}{2}}^r \tilde{F}'(s) \frac{s^{\delta-2}}{(z(s))^{\frac{1}{2}}} ds \right) \\ &\leq C \frac{n^2}{\sqrt{\mu_m}} F(r) \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)} \left(n^{-1} \mu_m^{-1} \tilde{F}\left(\frac{r}{2}\right) + n^{-\frac{7}{2}} \mu_m^{-\frac{3}{4}} \tilde{F}(r) r^{\delta-\frac{5}{2}} \right) \\ &\leq C \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)} \left(n e^{-\frac{nr}{2}} \mu_m^{-\frac{3}{2}} + n^{-\frac{3}{2}} \mu_m^{-\frac{5}{4}} r^{\delta-\frac{5}{2}} \right). \end{aligned} \quad (\text{A.44})$$

For $r \geq 1$, we obtain

$$\begin{aligned} n^2 |B_{mn}^3| &\leq C \frac{n^2}{\sqrt{\mu_m}} \frac{F(r)}{(1+z^2(r))^{\frac{1}{4}}} \left\{ \|f\|_{2,\alpha;Q}^{(1-\delta,2-\delta_0)} \mu_m^{-1} \int_0^{\frac{1}{2}} \frac{\tilde{F}(r)}{(1+z^2(r))^{\frac{1}{4}}} s^\delta ds \right. \\ &\quad \left. + \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)} n^{-2} \mu_m^{-1} \left(\int_{\frac{1}{2}}^1 \frac{\tilde{F}(r)}{(1+z^2(r))^{\frac{1}{4}}} s^{\delta-2} ds + \int_1^r \frac{\tilde{F}(r)}{(1+z^2(r))^{\frac{1}{4}}} s^{\delta_0-3} ds \right) \right\} \\ &\leq C \frac{n^2}{\sqrt{\mu_m}} F(r) \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)} \left\{ n^{-1} \mu_m^{-1} \int_0^{\frac{1}{2}} \tilde{F}'(s) ds + n^{-3} \mu_m^{-1} \left(\int_{\frac{1}{2}}^1 \tilde{F}'(s) \frac{s^{\delta-2}}{(z(s))^{\frac{1}{2}}} ds \right. \right. \\ &\quad \left. \left. + \int_1^r \tilde{F}'(s) \frac{s^{\delta_0-3}}{(z(s))^{\frac{1}{2}}} ds \right) \right\} \\ &\leq C \frac{n^2}{\sqrt{\mu_m}} F(r) \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)} \left(n^{-1} \mu_m^{-1} \tilde{F}\left(\frac{1}{2}\right) + n^{-\frac{7}{2}} \mu_m^{-\frac{3}{4}} \tilde{F}(1) + n^{-\frac{7}{2}} \mu_m^{-\frac{3}{4}} \tilde{F}(r) \right) \\ &\leq C \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)} \left(n e^{-\frac{n}{2}} \mu_m^{-\frac{3}{2}} + n^{-\frac{3}{2}} \mu_m^{-\frac{5}{4}} \right). \end{aligned} \quad (\text{A.45})$$

Combining (A.41)-(A.45) yields

$$\left\{ \begin{array}{l} \sum_{m>M} \sum_{\substack{n \geq 1 \\ 0 < r \leq L}} n^2 |R_{mn}^{(1)}(r)| \leq C \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)} r^{\min\{\delta, \frac{1}{2}\} - \frac{5}{2}}, \quad r \leq 1, \\ \sum_{m>M} \sum_{\substack{n \geq 1 \\ 0 < r \leq L}} n^2 |R_{mn}^{(1)}(r)| \leq C \|f\|_{4,\alpha;Q}^{(1-\delta,2-\delta_0)}, \quad 1 < r \leq L. \end{array} \right. \quad (\text{A.46})$$

Therefore, collecting (A.38)-(A.40) and (A.46) yields (A.37). \square

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